Quaternion-Kähler manifolds near maximal fixed point sets of S^1 -symmetries

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The classical Feix–Kaledin construction

B. Feix [5] (and D. Kaledin independently) showed that there exists a hyperkähler metric on a neighbourhood of the zero section of the cotangent bundle of any real-analytic Kähler manifold. B. Feix provided an explicit construction of its twistor space and showed that any hyperkähler manifold admitting a rotating circle action near its maximal fixed point set arises locally in this way.

- More generally:
  real-analytic type $(1, 1)$ complex connection on a complex manifold
  $\longleftrightarrow$ hypercomplex structure with rotating circle action

Question:

Can we obtain a similar result for quaternion-Kähler metric?

Quaternionic Feix–Kaledin construction

Theorem ($\sim$, Calderbank)

Let (S, J) be a $2n$-dimensional c-projective manifold such that the c-projective curvature is of type $(1,1)$. Let (L, ∇) be a holomorphic line bundle with a compatible complex connection with type $(1,1)$ curvature. We assume that all objects are real analytic. Then there exists a quaternionic structure on a neighbourhood of the zero section of $TS \otimes \mathcal{L}$, where $\mathcal{L}$ is some unitary line bundle constructed from L.

- As in the approach of Feix we explicitly constructed twistor space of the underlying quaternionic manifold.
- All quaternionic structures with circle action of the required type arise locally in this way (near a maximal fixed point set).

- For a fixed c-projective structure (e.g. Kähler manifold), by varying the line bundle, we obtain a family of quaternionic manifolds with rotating S^1-action!
- If in the c-pojective class there exists a connection with (full) type $(1,1)$ curvature, then there is a particular choice of (L,∇) which gives hypercomplex Feix's structure. If the connection is Kähler then the hypercomplex structure is hyperkähler.

When the obtained structures are quaternion-Kähler?

Theorem ($\sim$, Calderbank)

Let (S, J, g) be a Kähler-Einstein $2n$-manifold with the connection on $\mathcal{O}_S(1)$ induced by the Levi-Civita connection. Then the quaternionic Feix-Kaledin construction, with $\mathcal{L} = \mathcal{O}_S(k)$ a tensor power (possibly rational) of $\mathcal{O}_S(1)$, yields (locally) a quaternion-Kähler manifold.

- Is the Kähler-Einstein condition necessary?

No!

Definition

Let $n \geq 2$. A $4n$-dimensional smooth manifold M is called quaternionic if it is equipped with a rank 3 subbundle $Q \subset \mathrm{End}(TM)$ such that Q is fibrewise generated by three anti-commuting almost complex structures I, J, K satisfying

$$I^2 = J^2 = K^2 = IJK = -1,$$

and such that there exists a torsion-free connection D, called a quaternionic connection, preserving Q (i.e $D_X Q \subset Q$). A triple (M, g, Q) is called quaternion-Kähler if the metric g is Q-hermitian and its Levi-Civita connection is quaternionic.

A twistor space of a quaterninic manifold is the total space Z of the S^2-bundle of almost complex structures in Q. It turns out that there is a $1-1$ correspondence between twistor spaces and quaternionic manifolds. More precisely we have:

Quaternionic $4n$-manifolds	$\longleftrightarrow$	Complex (2n+1)-manifolds with a real structure τ and family of projective lines invariant under τ and with normal bundle $\mathbb{C}^{2n} \otimes \mathcal{O}(1)$.
Quaternion-Kähler $4n$-manifolds	$\longleftrightarrow$	Twistor spaces of quaternionic manifolds admitting a compatible holomorphic contact structure.
Hyperkähler $4n$-manifolds	$\longleftrightarrow$	Twistor spaces of quaternionic manifolds with holomorphic projection to $\mathbb{CP}^1$ and a holomorphic symplectic form along the fibres.

Main Theorem:

Theorem (∼, 2019)

1. *Let S be a real-analytic Kähler manifold, $\mathcal{L}$ a holomorphic line bundle on S with a connection $\nabla_{\mathcal{L}}$ such that the connection ∇ induced on $\mathcal{L} \otimes \mathcal{O}(1)$ is a unitary connection with curvature equal to $c\omega$, where ω is the Kähler form. Then the twistor space Z obtained by qFK from $(S, \omega, \mathcal{L}, \nabla_{\mathcal{L}})$ is a holomorphic contact manifold which is a twistor space of a quaternion-Kähler manifold M. Moreover the S^1-action on M coming from the construction is an isometry.*
2. *The scalar curvature of M equal to $2c$ provided that we fix the induced Kähler stucure on $S \subset M$ to be ω.*
3. *Any quaternion-Kähler manifold with an isometric rotating S^1 action near maximal fixed point set of the action arises locally in this way.*

Idea of the proof:

$$
\begin{array}{ccccc}
 & \mathcal{L}_{1,0}^* \otimes \mathcal{L}_{0,1} & \xrightarrow{\phi_{1,0}} & \mathcal{V}^{1,0} & \\
\swarrow & & & & \searrow \\
\mathbb{P}(\mathcal{L}_{1,0}^* \oplus \mathcal{L}_{0,1}^*) & & \overset{\phi}{\dashrightarrow} & & Z \\
\nwarrow & & & & \nearrow \\
 & \mathcal{L}_{1,0} \otimes \mathcal{L}_{0,1}^* & \xrightarrow{\phi_{0,1}} & \mathcal{V}^{0,1} &
\end{array}
\qquad (1)
$$

Standardization:

$$\phi_{1,0} \ : \ (p_1, \ldots, p_n, q_1, \ldots q_n, t) \mapsto (t, tp_1, \ldots, tp_n, q_1, \ldots q_n).$$

In this setting, along the leaves in the chosen direction, $(p_1, \ldots, p_n)$ are affine functions and the parallel sections are the constants (given by t).

We need to show that the push forward of the connection distribution extends to a contact distribution on neighbourhoods of the zero sections of $\mathcal{V}^{1,0}$ and $\mathcal{V}^{0,1}$.

- $\nabla = d + c\sum_{i=1}^{n} a_i(z,\overline{z})dz_i$, where $d\Sigma_{i=1}^{n} a_i dz_i = \Sigma_{i=1}^{n} da_i \wedge dz_i = \omega$.
- Unitarity: positive valued function $h(z,\overline{z})$ such that $c\sum_{i=1}^{n} a_i dz_i = \partial \log h = \sum_{i=1}^{n} \frac{\partial \log h}{\partial z_i} dz_i$.
  $\Longrightarrow \eta := \log h$ is a Kähler potential for ω as $d\partial = \overline{\partial}\partial$.
- After complexification the function $e^{-\eta(z,\tilde{z})} = [h(z,\tilde{z})]^{-1}$ is parallel along the leaves of the $(1,0)$-foliation
  $\Longrightarrow$ we may use it to obtain standardizing coordinates.

The connection distribution on $[\mathcal{L}_{1,0}]^* \otimes \mathcal{L}_{0,1}$ is given as the kernel of the following 1-form

$$\varphi := df - fc\Sigma_{i=1}^n a_i dz_i + cf\Sigma_{i=1}^n \tilde{a}_i d\tilde{z}_i,$$

where the $(1,0)$-foliation is by definition given by $\tilde{z} = const$,
$d\tilde{a}_i \wedge d\tilde{z}_i = -\omega$ and $\tilde{a}_i$ are affine along the leaves
$\Longrightarrow$ the standardizing coordinates $p_i := \tilde{a}_i$.
Setting $t := e^{-\eta} f$ we get

$$d[e^\eta t] - ce^\eta t\Sigma_{i=1}^n a_i dz_i + ce^\eta t\Sigma_{i=1}^n \tilde{a}_i d\tilde{z}_i = e^\eta[dt + ct\Sigma_{i=1}^n \frac{\partial \eta}{\partial \tilde{z}_i} d\tilde{z}_i + ct\Sigma_{i=1}^n \tilde{a}_i d\tilde{z}_i]$$

η is real valued on S hence $[\partial\eta](z,\overline{z}) = [\overline{\partial}\eta](\overline{z}, z)$ and therefore $\frac{\partial \eta}{\partial \tilde{z}_i} = \tilde{a}_i$.
$\Longrightarrow$ the form $e^{-\eta}\varphi$ extends to $\mathcal{V}^{1,0}$.

Remark

The contact form on a twistor space has values in a line bundle which after restriction to any of real twistor lines is isomorphic to $\mathcal{O}(2)$. In our setting this is represented by the fact that although the distribution given by φ extends to Z, the form φ does not. Indeed, the form $e^{-\eta}\varphi$ extends to one half of the twistor space (namely to $\mathcal{V}^{1,0}$) while $e^{\eta}\varphi$ extends to the other one ($\mathcal{V}^{0,1}$).

The converse:

For quaternion-Kähler manifold with a rotating symmetry the quaternion-Kähler moment map defines a distinguished (up to a sign) complex structure I. On the level of the twistor space it is defined by

$$\theta(\tilde{X}) = 0,$$

where θ is the contact form and $\tilde{X}$ is the vector field on the twistor space corresponding to the generator of the action (F. Battaglia, N.Hitchin).
In our setting this complex structure is an extension of the complex structure from the submanifold S.
Denote by $\mathcal{D}^{1,0} \subset \mathcal{V}^{1,0}$ the submanifold of the twistor Z corresponding to I.

One can show that $\mathcal{D}^{1,0}$ has the following properties:

- $\mathcal{D}^{1,0}$ is a rank n holomorphic vector subbundle of $\mathcal{V}^{1,0}$
- any fibre of $\mathcal{D}^{1,0}$ is tangent to the contact distribution on Z.

We already know that Z is a gluing of $\mathcal{V}^{1,0}$ and $\mathcal{V}^{0,1}$ which are bundles dual to the bundles of affine sections. Hence for any fibre $\tilde{x}$ the vector space $\mathcal{D}^{1,0}_{\tilde{x}}$ defines an affine section of the corresponding line bundle along the leaf $\tilde{x}$.

We declare these sections to be the parallel sections along the leaves of the $(1,0)$ foliation. Simiarly we obtain parallel sections along the leaves of the $(0,1)$ foliation and in this way were construct a distribution, hence the connection on $\mathbb{P}(\mathcal{L}^*_{1,0} \oplus \mathcal{L}^*_{0,1})$.

It can be shown that this connection satisfy the conditions we gave.

Gluing along the submanifold:

- If all structures are defined globally along S, we can construct a quaternion-Kähler metric on some tubular neighbourhood of the whole S.
- But we can do better!

The contact line bundle on the twistor space (i.e. the bundle in which the contact form takes the values) is the vertical bundle of the twistor fibration and along the zero section of $\mathcal{V}^{1,0}$ it is isomorphic to $\mathcal{L}_{1,0} \otimes [\mathcal{L}_{0,1}]^*|_S$, where S is the diagonal in $S^{\mathbb{C}}$.
Along the diagonal $\overline{\mathcal{L}_{1,0}} = \mathcal{L}_{0,1}$ and, as the bundle $\mathcal{L} \otimes \mathcal{O}(1)$ is unitary, we also have that $\mathcal{L}_{1,0} = \overline{\mathcal{L}_{1,0}}^* \implies$ The contact line bundle along the zero section is equal to $\mathcal{L}_{1,0}^{\otimes 2}$

It is necessary to assume that $\mathcal{L}_{1,0}^{\otimes 2}$ (with the corresponding connection on it) is globally defined on S
$\implies \mathcal{L} \otimes \mathcal{O}(1)$ may possibly be a square root bundle of a globally defined bundle.

The condition that $\mathcal{L} \otimes \mathcal{O}(1)$ is a square root bundle of a globally defined bundle is also sufficient for constructing a quaternion-Kähler metric globally along S!

- The complexified hermitian product on $\mathcal{L}^{\mathbb{C}} = \mathcal{L}^{1,0} \oplus \mathcal{L}^{0,1}$ over S^c gives a paring between $\mathcal{L}^{1,0}$ and $\mathcal{L}^{0,1}$ hence the bundle $\mathcal{L}^{1,0} \otimes \mathcal{L}^{0,1}$ is trivial and as a consequence the bundles $(\mathcal{L}_{1,0})^{\otimes k} \otimes (\mathcal{L}_{0,1})^{\otimes k}$ are canonically trivial for any k rational.
- The bundle $(\mathcal{L}_{1,0})^* \otimes \mathcal{L}_{0,1}$ is canonically isomorphic with $(\mathcal{L}_{0,1})^2$ everywhere.
- We glue globally the vector bundles $\mathcal{L}^2_{0,1}$ and $\mathcal{L}^2_{1,0}$ using canonical triviality of $\mathcal{L}_{1,0} \otimes \mathcal{L}_{0,1}$ in the following way:

$$\mathcal{L}^2_{0,1} \cong [(\mathcal{L}_{1,0})^* \otimes \mathcal{L}_{0,1}] \otimes [\mathcal{L}_{1,0} \otimes \mathcal{L}_{0,1}] \cong [(\mathcal{L}_{0,1})^* \otimes \mathcal{L}_{1,0}]^* \otimes [\mathcal{L}_{1,0} \otimes \mathcal{L}_{0,1}]^* \cong$$

$$\cong (\mathcal{L}^2_{1,0})^*.$$

Remarks:

- In case when S is compact, by uniqueness of Hermite-Einstein metrics on a holomorphic line bundle (unitary connection is Hermite-Einstein), the contact line bundle along S determines locally the quaternion-Kähler metric.
- The construction is strongly related with the hyperkähler/quaternion -Kähler correspondence – it describes it near the fixed point sets of the circle action.

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