Spherical CR-symmetric hypersurfaces in Hermitian symmetric spaces

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Almost CR-structure

Definition 1. ([Dragomir and Tomassini])

Let M be a $(2n-1)$-dimensional manifold and TM be its tangent bundle. An *almost CR-structure* on M is the subbundle $\mathcal{H} \subset \mathbb{C}TM = TM \otimes \mathbb{C}$ satisfying the following:

- each fiber $\mathcal{H}_p$, $p \in M$, is of complex dimension $n-1$
- $\mathcal{H} \cap \bar{\mathcal{H}} = \{0\}$, where $\bar{\mathcal{H}}$ denotes the complex conjugation of $\mathcal{H}$.

Then, such $(M; \mathcal{H})$ is said to be an almost CR manifold (of hypersurface type). In addition, if it satisfies

- $[\Gamma(\mathcal{H}), \Gamma(\mathcal{H})] \subset \Gamma(\mathcal{H})$ (integrability),

we call $(M; \mathcal{H})$ a CR manifold. Here, $\Gamma(\mathcal{H})$ denotes the space of all smooth sections on $\mathcal{H}$.

S. Dragomir and G. Tomassini, Differential geometry and analysis on CR manifolds. *Progr. Math.* 246, Birkhäuser Boston, Inc., Boston, MA, 2006.

• Given a CR manifold $(M; \mathcal{H})$, we have a unique subbundle $D = \text{Re}\{\mathcal{H} \oplus \bar{\mathcal{H}}\}$ of TM and a unique bundle isomorphism $J : D \to D$ such that $J^2 = -I$ and $\mathcal{H} = \{X - iJX | X \in \Gamma(D)\}$, where $\Gamma(D)$ denotes the space of all smooth sections of D. Such (D, J) is called the real representation of $\mathcal{H}$.

• Note that

$$[X, Y] - [JX, JY] \in \Gamma(D) \Leftrightarrow [\Gamma(\mathcal{H}), \Gamma(\mathcal{H})] \subset \Gamma(\mathbb{C}D)$$

(one calls it the partial integrability of $\mathcal{H}$),

$$[X, Y] - [JX, JY] + J[JX, Y] + J[X, JY] = 0$$

$$\Leftrightarrow [\Gamma(\mathcal{H}), \Gamma(\mathcal{H})] \subset \Gamma(\mathcal{H}).$$

• Assume the partial integrability of $\mathcal{H}$ holds and M is oriented, then it admits globally defined a nowhere zero section η, i.e., a real one-form on M such that $\mathrm{Ker}(\eta) = D$.

The corresponding Levi form L_η is defined by

$$L_\eta(X, Y) = -d\eta(X, JY), \quad X, Y \in \Gamma(D).$$

• If L_η is non-degenerate for some η, then the CR-structure (D, J) is said to be non-degenerate. Moreover, if L_η is hermitian, then it is said to be pseudo-Hermitian.
• If L_η is positive definite, then "non-degenerate" is replaced by "strongly pseudoconvex".

Definition 2.

For a fixed η, a non-degenerate (strongly pseudoconvex, resp.) integrable pseudo-Hermitian manifold $(M; \eta, J)$ is called a non-degenerate (strongly pseudoconvex, resp.) integrable CR manifold.

For a non-degenerate integrable CR manifold $M = (M; \eta, J)$, we have:

• There is a unique globally defined nowhere zero tangent vector field ξ on M such that $\eta(\xi) = 1$ and $d\eta(\cdot, \xi) = 0$. (We call it the *characteristic vector field* or *Reeb vector field* ([Reeb]).)
• Extend J to TM by ϕ: $\phi|_D = J$ and $\phi\xi = 0$.
• Extend the Levi-form to the *Webster metric* g_η on TM by $g_\eta = L_\eta + \eta \otimes \eta$, where $i_\xi L_\eta = 0$.

G. Reeb, Sur certaines propriétés topologiques des trajectories des systèmes dynamiques, Mémories de l'Acad. Roy, de Beligique, Sci. Ser. 2, 27, 1–62.

Definition 3. ([Tanaka],[Webster])

Tanaka-Webster connection $\hat{\nabla}$ on a non-degenerate integrable CR manifold $M = (M; \eta, J)$ is the unique linear connection satisfying the following conditions:

(i) $\hat{\nabla}\eta = 0$, $\hat{\nabla}\xi = 0$;
(ii) $\hat{\nabla}g_\eta = 0$, $\hat{\nabla}\phi = 0$;
$(iii-1)$ $\hat{T}(X, Y) = 2L_\eta(X, JY)\xi$, $X, \; Y \in \Gamma(D)$;
$(iii-2)$ $\hat{T}(\xi, \phi Y) = -\phi\hat{T}(\xi, Y)$, $Y \in \Gamma(D)$.

N. Tanaka, On non-degenerate real hypersurfaces, graded Lie algebras and Cartan connections, Japan J. Math. 2 (1976), 131–190.

S. M. Webster, Pseudohermitian structures on a real hypersurface, J. Diff. Geometry 13 (1978), 25–41.

Almost contact structure and the associated CR-structure

Definition 4. ([Gray],[Sasaki])

A $(2n-1)$-dimensional manifold M is said to be an *almost contact manifold* if its structure group of the linear frame bundle is reducible to $U(n-1)\times\{1\}$, or equivalently, if there exist a $(1,1)$-tensor field ϕ, a vector field ξ and a 1-form η satisfying

$$\eta(\xi)=1 \text{ and } \phi^2=-\mathrm{I}+\eta\otimes\xi, \tag{1}$$

where I denotes the identity transformation. We call (η,ξ,ϕ) an *almost contact structure*.

Gray, J.W. Some global properties of contact structure, *Ann. Math.* 69 (1959), 421–450.

Sasaki, S. On differentiable manifolds with certain structures which are closely related to almost contact structure I. *Tôhoku Math. J.* 12 (1960), 456–476.

For an almost contact manifold $M = (M; \eta, \xi, \phi)$, the tangent space T_pM of M at each point $p \in M$ is decomposed as $T_pM = D_p \oplus \{\xi\}_p$ (direct sum), where we denote $D_p = \{v \in T_pM | \eta(v) = 0\}$. Then $D : p \to D_p$ defines a distribution and the restriction $J = \phi|_D$ of ϕ to D defines an almost complex structure in D. Such (η, J) gives rise to an almost CR-structure.

Then we find:

• Given an almost contact structure, both *"non-degeneracy" of the Levi form* and *"CR-integrability" of the associated almost CR-structure* are not guaranteed, in general.

Normality

• Normality of almost contact structure: for an almost contact structure (η, ξ, ϕ) of M, the normality defined as follows. We may define naturally an almost complex structure J^x on $M \times \mathbb{R}$ by

$$J^x(X, f\frac{d}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt}),$$

where X is a vector field tangent to M, t the coordinate of $\mathbb{R}$ and f a function on $M \times \mathbb{R}$. If the almost complex structure J^x is integrable, M is said to be normal. Then we find that (cf. [Blair] p.92)

• The associated almost CR-structure of a normal almost contact manifold is CR-integrable.

On the other hand, one can find always a compatible Riemannian metric, namely which satisfies

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \quad (2)$$

for all vector fields X, Y on M. We call (η, ξ, ϕ, g) an *almost contact Riemannian structure* of M and $M = (M; \eta, \xi, \phi, g)$ an *almost contact Riemannian manifold*. From (1) and (2) we easily get

$$\phi\xi = 0, \quad \eta \circ \phi = 0, \quad \eta(X) = g(X, \xi). \quad (3)$$

Define a fundamental 2-form Φ by $\Phi(X, Y) = g(X, \phi Y)$.

• If $d\eta = \Phi$, then η is a contact structure, that is, $\eta \wedge (d\eta)^{n-1} \neq 0$. Then $M = (M; \eta, \xi, \phi, g)$ is called a *contact Riemannian manifold*.

• A normal contact Riemannian manifold is called a *Sasakian manifold*.

• For a contact Riemannian manifold M, we have the corresponding pseudo-Hermitian strongly pseudo-convex almost CR-structure with $g_\eta = g$.

• Tanno defined the generalized Tanaka-Webster connection by replacing the condition $\hat{\nabla}\phi = 0$ by $(\hat{\nabla}_X \phi)Y = \Omega(X, Y)$ on a contact Riemannian manifold (whose associated pseudo-Hermitian structure is not necessarily CR-integrable), where Ω is a $(1, 2)$-tensor field.

S. Tanno, Variational problems on contact Riemannian manifolds, Trans. Amer. Math. Soc., 314 (1989), 349-379.

For more details about the general theory of almost contact Riemannian manifolds, we refer to [Blair].

D. E. Blair, Riemannian geometry of contact and symplectic manifolds, Progress in Math. 203, Birkhäuser, Boston, Basel, Berlin, 2010.

Real hypersurfaces of Kähler manifolds

Let M be an oriented real hypersurface of a Kähler manifold $\widetilde{M} = (\widetilde{M}; \widetilde{J}, \widetilde{g})$ and N a global unit normal vector field on M. By $\widetilde{\nabla}$, S we denote the Levi-Civita connection in $\widetilde{M}$ and the shape operator with respect to N, respectively.

Then the Gauss and Weingarten formulas are given respectively by

$$\widetilde{\nabla}_X Y = \nabla_X Y + g(SX, Y)N, \quad \widetilde{\nabla}_X N = -SX$$

for any vector fields X and Y tangent to M, where g denotes the Riemann metric of M induced from $\widetilde{g}$. An eigenvector(resp. eigenvalue) of the shape operator S is called a principal curvature vector(resp. principal curvature).

For any vector field X tangent to M, we put

$$\widetilde{J}X = \phi X + \eta(X)N, \quad \widetilde{J}N = -\xi. \tag{4}$$

We easily see that the structure (η, ξ, ϕ, g) is an almost contact metric structure on M i.e. satisfies (1) and (2).

From the condition $\widetilde{\nabla}\widetilde{J} = 0$, the relations (4) and by making use of the Gauss and Weingarten formulas, we have

$$(\nabla_X \phi)Y = \eta(Y)SX - g(SX, Y)\xi, \quad (5)$$
$$\nabla_X \xi = \phi SX. \quad (6)$$

By using (5) and (6), we see that a real hypersurface in a Kähler manifold always satisfies the CR-integrability condition.

Proposition 1.

A real hypersurface of a Kähler manifold admits an integrable CR-structure.

But, non-degeneracy of the Levi-form still remains not to be guaranteed.

A real hypersurface of a Kähler manifold is called a *contact hypersurface* if M satisfies $d\eta = \rho\Phi$, $\rho \neq 0$ (due to [Okumura]). Then we find that the Levi-form of contact hypersurfaces is positive-definite (or negative-definite).

M. Okumura, Contact hypersutfaces in certain Kaehlerian manifolds, Tôhoku Math. J. 18(2) (1966), 74-102.

Here, we note

• Let $M = (M; \eta, \phi, \xi, g)$ be a real hypersurface of a Kähler manifold. The almost contact metric structure of M is contact metric if and only if $\phi A + A\phi = \pm 2\phi$.

Theorem 2.([Okumura], [Kon], [Vernon], [Suh], [Adachi-Kameda-Maeda])

Let M be a real hypersurface of a complex space form $\widetilde{M}_n(c)$. Then M is contact if and only if M is locally congruent to one of the following:

(I) in case that $\widetilde{M}_n(c) = P_n\mathbb{C}$ (with Fubini-Study metric),
(A_1) a geodesic hypersphere of radius r, where $0 < r < \frac{\pi}{\sqrt{c}}$,
(B) a tube of radius r over a complex quadric Q_{n-1}, where $0 < r < \frac{\pi}{2\sqrt{c}}$;

(II) in case that $\widetilde{M}_n(c) = H_n\mathbb{C}$ (with Bergman metric),

(A_0) a horosphere,
(A_1) a geodesic hypersphere or a tube over a complex hyperbolic hyperplane $H_{n-1}\mathbb{C}$,
(B) a tube over a totally real hyperbolic space $H_n\mathbb{R}$

(III) in case that $\widetilde{M}_n(c) = E_n\mathbb{C}$,

(A) $S^{2n-1}(r)$,
(B) $S^{n-1}(r) \times \mathbb{R}^n$.

- M. Kon, Pseudo-Einstein real hypersurfaces of complex space forms, J. Diff. Geometry 14 (1979), 339-354.
- M. H. Vernon, Contact hypersutfaces in complex hyperbolic space, Tôhoku Math. J. 39 (1987), 215-222.
- Y.J. Suh, On real hypersurfaces of a complex space form with η-parallel Ricci tensor, Tsukuba J. Math. 14 (1990), 27-37.
- T. Adachi, M. Kameda and S. Maeda, Real hypersurfaces which are contact in a nonflat complex space form, Hokkaido Math. J. 40 (2011), 205-217.

The complex quadrics and the noncompact dual spaces

• The homogeneous quadratic equation $z_1^2 + \cdots + z_{n+2}^2 = 0$ on $\mathbb{C}^{n+2}$ defines a complex hypersurface Q^n in the $(n+1)$-dimensional complex projective space $\mathbb{C}P^{n+1}$.

• The complex quadric $Q^n = SO_{n+2}/SO_nSO_2$ admits the Kähler structure $(\widetilde{J}, \widetilde{g})$ which is induced from $\mathbb{C}P^{n+1}$.

• Another geometric structure is a rank two vector bundle $\mathcal{A}$ over Q^n which contains an S^1-bundle of real structures on the tangent spaces of Q^n. The bundle $\mathcal{A}$ is just the family of shape operators with respect to the normal vectors in the rank two normal bundle.

Then

• Gauss equation for $Q^n \subset \mathbb{C}P^{n+1}$ implies that the Riemann curvature tensor can be expressed by the Kähler structure $(\widetilde{J}, \widetilde{g})$ and a generic real structure A in $\mathcal{A}$.

• For a real structure $A \in \mathcal{A}$ we denote by $V(A)$ its $\frac{\sqrt{c}}{2}$-eigenspace; then $\widetilde{J}V(A)$ is the $-\frac{\sqrt{c}}{2}$-eigenspace of A, where c denotes the maximal sectional curvature $c > 0$.

• There are two types of singular tangent vectors: (i) if there exists a real structure $A \in \mathcal{A}$ such that $W \in V(A)$, then W is singular. Such a singular tangent vector is called $\mathcal{A}$-principal. (ii) if there exist a real structure $A \in \mathcal{A}$ and orthonormal vectors $X, Y \in V(A)$ such that $W/||W|| = (X + \widetilde{J}Y)/\sqrt{2}$, then W is singular. Such a singular tangent vector is called $\mathcal{A}$-isotropic.

We should remark

• The noncompact dual space $Q^{n*} = SO^{o}_{n,2}/SO_nSO_2$ is not realized as a homogeneous complex hypersurface in the $(n+1)$-dimensional complex hyperbolic space $\mathbb{C}H^{n+1}$.

For more details about the geometric structure of Q^n or Q^{n*} and the fundamental properties of their real hypersurfaces, we refer to [BerndtSuh3,4], [KleinSuh].

J. Berndt and Y.J. Suh, Real hypersurfaces with isometric Reeb flow in complex quadrics, Internat. J. Math. 24 (2013), 1350050 (18 pages).

S. Klein and Y.J. Suh, Contact real hypersurfaces in the complex hyperbolic quadric, Ann. di Mat. Pure Appl. (2019).

Due to a result of J. Berndt and Y.J. Suh and a result independently done by T. H. Loo, we have the following theorem.

Theorem 3.

(I) Let M be a real hypersurface in the complex quadric Q^n and $n \geq 3$. Then M is a contact hypersurface if and only if M is locally congruent to a tube of radius $0 < r < \pi/\sqrt{2c}$ around a real form S^n of Q^n.

(II) Let M be a real hypersurface in the noncompact dual Q^{n*} of the complex quadric and $n \geq 3$. Then M is a contact hypersurface if and only if M is locally congruent one of the following:

(1) *a tube of radius* $r \in \mathbb{R}_+$ *around the totally geodesic* $Q^{(n-1)*}$ *in* Q^{n*},
(2) *a horosphere in* Q^{n*} *whose center at infinity is determined by an* $\mathcal{A}$-*principal geodesic in* Q^{n*},
(3) *tube of radius* $r \in \mathbb{R}_+$ *around a real form* $\mathbb{R}H^n$ *in* Q^{n*}.

J. Berndt and Y.J. Suh, Contact hypersurfaces in Kähler manifolds, Proc. Amer. Math. Soc. 142 (2014), 2637-2649.

T.H. Loo, $\mathfrak{A}$-Hopf hypersurfaces in complex quadrics, preprint (arXiv:1712.00538 [math.DG]).

Contact (k, μ)-spaces

According to [BKP], a contact Riemannian manifold M is said to be a *contact* (k, μ)*-space* if there exist $(k, \mu) \in \mathbb{R}^2$ such that

$$R(X, Y)\xi = (kI + \mu h)(\eta(Y)X - \eta(X)Y)$$

for all $X, Y \in \Gamma(TM)$, where $2h = L_\xi \phi$. Then we have

- $k \leq 1$ and the structure is Sasakian $(h = 0)$ when $k = 1$,
- the associated pseudo-Hermitian structure is CR-integrable.

D. E. Blair, T. Koufogiorgos and B. J. Papantoniou, Contact metric manifolds satisfying a nullity condition, Israel J. Math. 91 (1995), 189-214.

Classification

We have a fundamental theorem.

Theorem 4. (E. Boeckx)

Let $(M^{2n+1};\eta,\xi,\phi,g)$ and $(M'^{2n+1};\eta',\xi',\phi',g')$ be two non-Sasakian (k,μ)-spaces. Then they are locally isometric as contact metric spaces. In particular, if both spaces are simply connected and complete, they are globally isometric up to a pseudo-homothetic transformation.

Recall the *gauge transformation* (or *pseudo-conformal transformation*) f of a contact metric manifold (pseudo-Hermitian strongly pseudo-convex almost CR manifold, respectively) M. Given a contact form η, we consider a new contact form $f^*\eta = a\eta$ for a positive function a. By assuming $J \circ f_* = f_* \circ J$, then the associated structures are determined in natural way (Lemma 9.1 in [Tanno]):

$$f_*\xi = \frac{1}{a}(\xi + \zeta),\ \zeta = \frac{1}{2a}\phi(\text{grad } a),\ \phi \circ f_* = f_* \circ \phi + \frac{1}{2a}\eta \otimes (\text{grad } a - (\xi a)\xi),$$
$$f^*g = ag - a(\eta \otimes \nu + \nu \otimes \eta) + a(a - 1 + \|\zeta\|^2)\eta \otimes \eta,$$

where ν is dual to ζ with respect to g.

Note that:

• $\bar{g} = f^*g$ and g are conformally related for D.
• If a is constant, then f reduces to a D_a-*homothetic* or *pseudo-homothetic transformation*:
$f^*\eta = a\eta,\ f_*\xi = \frac{1}{a}\xi,\ \phi \circ f_* = f_* \circ \phi,\ f^*g = ag + a(a-1)\eta \otimes \eta.$
• For $a = 1$, it includes an isometric (actually authomorphic) transformation.

Then we have

- the pseudo-homothetic deformation of a contact (k,μ)-space is another contact $(\bar{k},\bar{\mu})$-space with $\bar{k}=(k+a^2-1)/a^2$ and $\bar{\mu}=(\mu+2a-2)/a$.
- $k=1$ and $\mu=2$ are two invariants under pseudo-homothetic deformation for all $a\neq 1$.
- The *Boeckx invariant* $\mathcal{I}$ of a *non-Sasakian* contact (k,μ)-space M is defined by $\mathcal{I}=(1-\mu/2)/\sqrt{1-k}$, which determines locally a contact (k,μ)-space, up to equivalence.

Theorem 5. (E. Boeckx)

Let $(M_i;\eta_i,\xi_i,\phi_i,g_i)$, $i=1,2$, be two contact (k_i,μ_i)-spaces of the same dimension. Then $\mathcal{I}_{M_1}=\mathcal{I}_{M_2}$ if and only if, up to a pseudo-homothetic deformation of the contact metric structure, the two spaces are locally equivalent as contact metric spaces. In particular, if both spaces are simply connected and complete, they are globally isometric up to a pseudo-homothetic transformation.

E. Boeckx, A full classifiation of contact metric (k,μ)-spaces, Illinois J. Math. 44 (2000), 212-219.

Then we have

Theorem 6. ([Cho4])

The simply-connected, complete, non K-contact, contact Riemannian space M is a (k, μ)-space if and only if it is equivalent (up to a pseudo-homothetic transformation) to one of (i), (ii), (iii), (iv), and (v) in the following table:

	real hypersurfaces	max(min)curv.	Boeckx inv.
(i)	a tube of radius $r=\sqrt{\frac{2}{c}}\arctan\frac{2\sqrt{2}}{\sqrt{c}}\in(0,\pi/\sqrt{2c})$ around a real form S^n of Q^n	$c>0$	$\mathcal{I}>1$
(ii)	$\mathbb{R}^n\times S^{n-1}(1/2)$ in $\mathbb{C}^n$	$c=0$	$\mathcal{I}=1$
(iii)	a tube of radius $r=\sqrt{\frac{2}{\lvert c\rvert}}\coth^{-1}\frac{2\sqrt{2}}{\sqrt{\lvert c\rvert}}$ around a real form $\mathbb{R}H^n$ in Q^{n*}	$-4<c<0$	$0<\mathcal{I}<1$
	a tube of radius $r=\frac{\sqrt{2}}{2}\coth^{-1}\sqrt{2}$ around a real form $\mathbb{R}H^n$ in Q^{n*}	$c=-4$	$\mathcal{I}=0$
	a tube of radius $r=\sqrt{\frac{2}{\lvert c\rvert}}\coth^{-1}\frac{2\sqrt{2}}{\sqrt{\lvert c\rvert}}$ around a real form $\mathbb{R}H^n$ in Q^{n*}	$-8<c<-4$	$-1<\mathcal{I}<0$
(iv)	a horosphere in Q^{n*} whose center at infinity is determined by an $\mathcal{A}$-principal geodesic in Q^{n*}	$c=-8$	$\mathcal{I}=-1$
(v)	a tube of radius $r=\sqrt{\frac{2}{\lvert c\rvert}}\tanh^{-1}\frac{2\sqrt{2}}{\sqrt{\lvert c\rvert}}$ around Q^{n-1*} in Q^{n*}	$c<-8$	$\mathcal{I}<-1$

Remarks. The case of Boeckx invariant $\mathcal{I} = 1, 0, -1$, respectively has the following remarkable characterization.

1. Due to the result of Boeckx and Cho [BoeckxCho2] the case (ii) $\mathbb{R}^n \times S^{n-1}(4)$ in $\mathbb{C}^n$ is the unique non-Sasakian simpy connected Riemannian symmetric contact metric manifold.

2. In the case (iii), a tube of radius $r = \frac{\sqrt{2}}{2}\log(\sqrt{2}+1)$ around a real form $\mathbb{R}H^n$ in Q^{n*} has specifically a nice CR-geometric property: it is spherical ([Cho2]), that is, it is locally CR-equivalent to the sphere S^{2n-1} endowed with the standard CR structure as a real hypersurface of $\mathbb{C}^n$, and moreover it has constant holomorphic sectional curvature $\hat{H} = 0$ with respect to the Tanaka-Webster connection $\hat{\nabla}$ ([Cho1],[Cho3]). Also, it is weakly η-Einstein ([ChoChunEuh]).

3. Other than the rigid Ricci soliton $\mathbb{R}^n \times S^{n-1}(4)$ in $\mathbb{C}^n$ (the case (ii)), the case (iv) a horosphere in Q^{n*} is only a non-Sasakian simply-connected complete Ricci soliton in the class of contact (k, μ)-spaces, $n \geq 3$ ([CHKTT]).

- J.T. Cho, Geometry of contact strongly pseudo-convex CR-manifolds, J. Korean Math. Soc. 43 (2006), 1019-1045.
- J.T. Cho, Contact Riemannian manifolds with vanishing Gauge invariant, Toyama Math. J. 31 (2008), 1-16.
- J.T. Cho, Strongly pseudo-convex CR space forms, Complex Manifolds 6 (2019), 279-293.
- J.T. Cho, Contact hypersurfaces and CR-symmetry, Ann. Mat. Pura Appl. 199 (2020), 1873-1884.
- J.T. Cho, T. Hashinaga, A. Kubo, Y. Taketomi, H. Tamaru, Realizations of some contact metric manifolds as Ricci soliton real hypersurfaces. J. Geom. Phys. 123 (2018), 221-234.
- E. Boeckx and J.T. Cho, Locally symmetric contact metric manifolds, Monatsh. Math. 148 (2006), 269-281.
- J.T. Cho, S.H. Chun, Yunhee Euh, Weakly η-Einstein contact manifolds, Results Math. 77 (2022), 110, 16pp.

Recall

Definition 5. ([Kaup-Zaitsev])

Let $(M; \eta, J)$ be a pseudo-Hermitian manifold with almost CR-structure (η, J). Then a Webster metric g_η is CR-symmetric if for each point $p \in M$ there exists an isometric CR-diffeomorphism $\sigma : M \to M$ such that

$$\sigma_p(p) = p, \ (d\sigma)_p(X) = -X \text{ for all } X \in D_p.$$

Notes.

1. Since the symmetry at p is uniquely determined, we may define the local version in a natural way.
2. A (locally) CR-symmetric pseudo-Hermitian almost CR manifold is automatically CR integrable because the symmetries are CR maps.
3. An isometric CR-diffeomorphism implies an automorphism of the corresponding contact Riemannian structure (η, ξ, ϕ, g) (cf. [Dileo-Lotta]), indeed, the differential of σ_p at p is given by

$$(d\sigma_p)_p = -id + 2\eta_p \otimes \xi_p.$$

Theorem 7. ([Dileo-Lotta])

Let $(M;\eta,J)$ be a pseudo-Hermitian manifold whose Webster metric g_η is not Sasakian. Then, the Webster metric g_η is locally CR-symmetric if and only if the underlying contact Riemannian manifold is a (k,μ)-space.

In proving this result, the following characterization of contact (k,μ)-spaces makes an essential role.

Theorem 8. ([Boeckx-Cho])

A non K-contact, contact Riemannian space M is a (k,μ)-space if and only if h is η-parallel:

$$g((\nabla_X h)Y,Z)=0$$

for all $X,Y,Z\in\Gamma(D)$.

W. Kaup and D. Zaitsev, On symmetric Cauchy-Riemann manifolds, Adv. Math. 149 (2000), 145-181.

G. Dileo and A. Lotta, A classification of spherical symmetric CR manifolds, Bull. Austral. Math. Soc. 80 (2009), 251-274.

E. Boeckx and J.T. Cho, η-parallel contact metric spaces, Differential Geom. Appl. 22 (2005), 275–285.

For a Sasakian manifold, its associated pseudo-Hermitian CR structure is (locally) CR-symmetric if and only if it is (locally) ϕ-symmetric space in the sense of Takahashi ([DileoLotta]). We have

Corollary 1. ([Cho4])

A contact strongly pseudo-convex almost CR manifold $M = (M; \eta, J)$ is (locally) CR-symmetric if and only if M is either a Sasakian (locally) ϕ-symmetric space or (locally) equivalent (up to a pseudo-homothetic transformation) to one of (i), (ii), (iii), (iv) and (v) in the table of Theorem 9.

Definition 6. ([BoeckxCho3])

Let $(M; \eta, J)$ be a contact strongly pseudo-convex almost CR manifold. Then M is said to be a locally pseudo-Hermitian symmetric space if all characteristic $\hat{\nabla}$-reflections are local affine mappings, i.e., they preserve the generalized Tanaka-Webster connection $\hat{\nabla}$. If any such characteristic $\hat{\nabla}$-reflection is extendable as global affine transformation (with respect to $\hat{\nabla}$), then we call M to be a globally pseudo-Hermitian symmetric space.

Then we proved that a contact strongly pseudo-convex almost CR manifold is locally pseudo-Hermitian symmetric space if and only if it is either a Sasakian locally ϕ-symmetric space or a non-Sasakian (k, μ)-space ([Theorem 14, BoeckxCho3]). Thus we have

Corollary 2.

A contact strongly pseudo-convex almost CR manifold $M = (M; \eta, J)$ is locally pseudo-Hermitan symmetric if and only if M is either a Sasakian locally ϕ-symmetric space or locally equivalent (up to a pseudo-homothetic transformation) to one of (i), (ii), (iii), (iv) and (v) in the table of Theorem 6.

Assume that $(M;\eta,J)$ is complete and simply connected. Then, since a complete and simply connected Sasakian locally ϕ-symmetric space is a globally ϕ-symmetric space, at last we have

Corollary 3.

A complete and simply connected contact strongly pseudo-convex almost CR manifold $M=(M;\eta,J)$ is (globally) pseudo-Hermitan symmetric if and only if M is either a Sasakian (globally) ϕ-symmetric space or equivalent (up to a pseudo-homothetic transformation) to one of (i), (ii), (iii), (iv) and (v) in the table of Theorem 6.

E. Boeckx and J.T. Cho, Pseudo-Hermitian symmetries, Israel J. Math. 166 (2008), 125-145.

Spherical CR manifolds

• A *spherical* CR manifold is a contact strongly pseudo-convex CR manifold which is locally CR-equivalent to the sphere S^{2n+1} endowed with the standard CR structure as a real hypersurface of $\mathbb{C}^{n+1}$.
• Spherical CR manifolds are characterized by $C = 0$, where C is the Chern-Moser-Tanaka (pseudo-conformal) invariant tensor of type $(1, 3)$.
• The simply connected homogeneous spherical hypersurfaces of the Euclidean space $\mathbb{C}^{n+1}$ were classified by Burns and Shnider.

S. S. Chern and J. K. Moser, Real hypersurfaces in complex manifolds, Acta Math. 133 (1974), 219–271.

D. Burns and S. Shnider, Spherical hypersurfaces in complex manifolds, Inven. Math. 33 (1976), 223-246.

The following theorem was proved in [DileoLotta].

Theorem 9.

Every complete and simply connected spherical CR-symmetric manifold M^{2n+1}, $n \geq 2$, is equivalent up to a pseudo-homothetic diffeomorphism, to one of the following spaces: S^{2n+1}, $Heis^{2n+1}$, $B^n \times \mathbb{R}$, $P_1^n, \cdots, P_{n-1}^n$, $T_1 H^{n+1}(-1)$.

G. Dileo and A. Lotta, A classification of spherical symmetric CR manifolds, Bull. Austral. Math. Soc. 80 (2009), 251-274.

The $n-1$ spaces P^n_k, $k = 1, \cdots, n-1$, denote the total space of a principal fiber bundle $P^n_k \to \mathbb{C}P^k \times \mathbb{C}H^{n-k}$, where the base space is the product of complex projective space $\mathbb{C}P^k$ with positive constant holomorphic sectional curvature $c(>0)$ and complex hyperbolic space $\mathbb{C}H^{n-k}$ with negative constant holomorphic sectional curvature $-c$. Those P^n_k's are not only Sasakian ϕ-symmetric (due to [JimenezKowalski]) but they are spherical since the base manifold is Bochner flat (see [Bryant], [MatsumotoTanno]).

- J. A. Jiménez and O. Kowalski, The classification of ϕ-symmetic Sasakian manifolds, Monatsh. Math. 115 (1993), 83-98.
- R. L. Bryant, Bochner-Kähler metrics, J. Amer. Math. Soc. 14 (2001), 623-715.
- M. Matsumoto and S. Tanno, Kählerian spaces with parallel or vanishing Bochner curvature tensor, Tensor (N.S.) 27 (1973), 291-294.

We have the following facts:

• Since a geodesic sphere $G^{2k+1}(r)$ is a tube over a totally geodesic $\mathbb{C}P^k$ in $\mathbb{C}P^{k+1}$, $G^{2k+1}(r)$ is a total space of a circle bundle over $\mathbb{C}P^k$, and hence $G^{2k+1}(\frac{\pi}{4}) \times \mathbb{C}H^{n-k}(-4)(\subset \mathbb{C}P^{k+1}(4) \times \mathbb{C}H^{n-k}(-4)$, as a real hypersurface) is also regarded as a total space of a circle bundle over $\mathbb{C}P^k(4) \times \mathbb{C}H^{n-k}(-4)$.

• For a tube $T^{2k+1}(r)$ of radius r around $\mathbb{C}H^k$ in $\mathbb{C}H^{k+1}$, we have a principal fiber bundle $T^{2k+1}(r) \times \mathbb{C}P^{n-k} \to \mathbb{C}H^{k+1} \times \mathbb{C}P^{n-k}$ with the structure group S^1. In particular, $T^{2k+1}((\frac{\sqrt{2}}{2}\ln(\sqrt{2}+1)) \times \mathbb{C}P^{n-k}(8)$ is realized as a real hypersurface of Hermitian symmetric space $\mathbb{C}H^{k+1}(-8) \times \mathbb{C}P^{n-k}(8)$, which are Sasakian ϕ-symmetic and at the same time spherical.

Then, we have the realizations of spherical CR-symmetric spaces as real hypersurfaces in Hermitian symmetric spaces.

Theorem 10.

Every complete and simply connected spherical CR-symmetric manifold M^{2n+1}, $n \geq 2$, is either isometric to one of the following real hypersurfaces in complex manifolds:

- (i) a geodesic hypersphere $G^{2n+1}(r)$ with $r = \frac{2}{\sqrt{c}} \arctan(\frac{\sqrt{c}}{2})$ in $\mathbb{C}P^{n+1}(c)$ $(c > 0)$ (with constant ϕ-sectional curvature $H > 1$),
- (ii) a unit sphere $S^{2n+1}(1)$ in $\mathbb{C}^{n+1}(0)$ (with constant ϕ-sectional curvature $H = 1$),
- (iii) a geodesic hypersphere $G^{2n+1}(r)$ with $r = \frac{2}{\sqrt{-c}} \tanh^{-1}(\frac{\sqrt{-c}}{2})$ in $\mathbb{C}H^{n+1}(c)$ $(-4 < c < 0)$ (with $-3 < H < 1$),
- (iv) a horosphere in $\mathbb{C}H^{n+1}(-4)$ (of constant holomorphic sectional curvature -4) (with $H = -3$),
- (v) a tube $T^{2n+1}(r)$ of radius $r = \frac{2}{\sqrt{-c}} \coth^{-1}(\frac{\sqrt{-c}}{2})$ around a totally geodesic $\mathbb{C}H^{n}(c)$ in $\mathbb{C}H^{n+1}(c)$ $(c < -4)$ (with $H < -3$),

or equivalent, up to a pseudo-homothetic diffeomorphism, to

(*vi*) a real hypersurface $G^{2k+1}(\frac{\pi}{4}) \times \mathbb{C}H^{n-k}(-4)$ in $\mathbb{C}P^{k+1}(4) \times \mathbb{C}H^{n-k}(-4)$, where $G^{2k+1}(r)$ denotes a tube of radius r around totally geodesic $\mathbb{C}P^k$ in $\mathbb{C}P^{k+1}(4)$, $k = 1, 2, \cdots, n-1$,

(*vii*) a real hypersurface $T^{2k+1}(\frac{\sqrt{2}}{2}\ln(\sqrt{2}+1)) \times \mathbb{C}P^{n-k}(8)$ in $\mathbb{C}H^{k+1}(-8) \times \mathbb{C}P^{n-k}(8)$, where $T^{2k+1}(r)$ denotes a tube of radius r around torally geodesic $\mathbb{C}H^k$ in $\mathbb{C}H^{k+1}(-8)$, $k = 1, 2, \cdots, n-1$;

(*viii*) a tube of radius $r = \frac{\sqrt{2}}{2}\ln(\sqrt{2}+1)$ around totally geodesic $\mathbb{R}H^{n+1}$ in the non-compact dual of complex quadric $Q^{n+1*}(-4)$ of minimal curvature -4.

J. T. Cho and M. Kimura, Spherical CR-symmetric hypersurfaces in Hermitian symmetric spaces, preprint.

CR-symmetric contact 3-manifolds

• Every 3-dimensional contact strongly pseudo-convex CR manifold is spherical.
• Blair and Vanhecke proved that a three-dimensional complete and simply connected Sasakian locally ϕ-symmetric space occurs only in Sasakian space forms.

Then, together with Corollary 1, we have

Theorem 11.

A complete and simply connected CR-symmetric contact 3-manifold M is either isometric to one of the following real hypersurfaces in Hermitian symmetric surfaces:

(i) a geodesic hypersphere $G^3(r)$ with $r = \frac{2}{\sqrt{c}} \arctan(\frac{\sqrt{c}}{2})$ in $\mathbb{C}P^2(c)$ $(c > 0)$,

(ii) a unit sphere $S^3(1)$ in $\mathbb{C}^2(0)$,

(iii) a geodesic hypersphere $G^3(r)$ with $r = \frac{2}{\sqrt{-c}} \tanh^{-1}(\frac{\sqrt{-c}}{2})$ in $\mathbb{C}H^2(c)$ $(-4 < c < 0)$,

(iv) a horosphere in $\mathbb{C}H^2(-4)$,

(v) a tube $T^3(r)$ of radius $r = \frac{2}{\sqrt{-c}} \coth^{-1}(\frac{\sqrt{-c}}{2})$ around a totally geodesic $\mathbb{C}H^1(c)$ in $\mathbb{C}H^2(c)$ $(c < -4)$;

or (globally) equivalent, up to a pseudo-homothetic diffeomorphism, to one of the following real hypersurfaces in Hermitian symmetric surfaces:

(*vi*) a tube of radius $r = \sqrt{\frac{2}{c}} \arctan \frac{2\sqrt{2}}{\sqrt{c}}$ around a real form S^2 of $Q^2(c)$, where $c > 0$ is maximal curvature of Q^2,

(*vii*) $\mathbb{R}^2 \times S^1(\frac{1}{2})$ in $\mathbb{C}^2(0)$,

(*viii*) a tube of radius $r = \sqrt{\frac{2}{|c|}} \coth^{-1} \frac{2\sqrt{2}}{\sqrt{|c|}}$ around totally a geodesic $\mathbb{R}H^2$ in the non-compact dual of complex quadric $Q^{2*}(c)$, where $-8 < c < 0$ is minimal curvature,

(*ix*) a horosphere in $Q^{2*}(-8)$ whose center at infinity is determined by an $\mathcal{A}$-principal geodesic in $Q^{2*}(-8)$,

(*x*) a tube of radius $r = \sqrt{\frac{2}{|c|}} \tanh^{-1} \frac{2\sqrt{2}}{\sqrt{|c|}}$ around a totally geodesic Q^{1*} in $Q^{2*}(c)$, where minimal curvature $c < -8$.

Here, the above cases (i) - (v) are Sasakian, and the other cases (vi) - (x) are non-Sasakian.

Thank you for your attention !!