Sasakian geometry on sphere bundles

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Outline

Sasakian Geometry:

Sasakian structure on a smooth manifold M of dimension $2n+1$

A quadruple $\mathcal{S} = (\xi, \eta, \Phi, g)$ where

- η is contact 1-form defining a subbundle (contact bundle) in TM by $\mathcal{D} = \ker \eta$.
- ξ is the Reeb vector field of η [$\eta(\xi) = 1$ and $\xi \rfloor d\eta = 0$]
- Φ is an endomorphism of the tangent bundle TM such that $\Phi^2 = -Id + \xi \otimes \eta$
- $J = \Phi|_{\mathcal{D}}$ is a complex structure compatible with the symplectic form $d\eta$ on $\mathcal{D}$.
- $g(\cdot,\cdot) := d\eta(\Phi\cdot,\cdot) + \eta(\cdot)\eta(\cdot)$ is a Riemannian metric
- ξ is a Killing vector field of g which generates a one dimensional foliation $\mathcal{F}_\xi$ of M whose transverse structure is Kähler: (g_T, ω_T).
- The cone metric, $(dt^2 + t^2 g, d(t^2\eta))$, is Kähler on $C(M) = M \times \mathbb{R}^+$ with complex structure I: $IY = \Phi Y + \eta(Y)t\frac{\partial}{\partial t}$ for vector fields Y on M, and $I(t\frac{\partial}{\partial t}) = -\xi$.

Transverse Kähler structure

- If ξ is regular, the transverse Kähler structure lives on a smooth manifold (quotient of regular foliation $\mathcal{F}_\xi$).
- If ξ is quasi-regular, the transverse Kähler structure has orbifold singularities (quotient of quasi-regular foliation $\mathcal{F}_\xi$).
- If not regular or quasi-regular we call it irregular... (that's most of them)

Transverse Homothety:

- If $\mathcal{S} = (\xi, \eta, \Phi, g)$ is a Sasakian structure, so is $\mathcal{S}_a = (a^{-1}\xi, a\eta, \Phi, g_a)$ for every $a \in \mathbb{R}^+$ with $g_a = ag + (a^2 - a)\eta \otimes \eta$.
- So Sasakian structures come in rays.

Deforming the Sasaki structure:

In its contact structure isotopy class:

- $\eta \to \eta + d^c\phi, \quad \phi$ is basic
- This corresponds to a deformation of the transverse Kähler form $\omega_T \to \omega_T + dd^c\phi$ in its Kähler class in the regular/quasi-regular case.
- "Up to isotopy" means that the Sasaki structure might have to been deformed as above.

In the Sasaki Cone:

- Choose a maximal torus T^k, $0 \leq k \leq n+1$ in the Sasaki automorphism group $\mathfrak{Aut}(\mathcal{S})$
- The unreduced Sasaki cone is $\mathfrak{t}^+ = \{\xi' \in \mathfrak{t}_k \,|\, \eta(\xi') > 0\}$, where $\mathfrak{t}^k$ denotes the Lie algebra of T^k.
- Each element in $\mathfrak{t}^+$ determines a new Sasaki structure with the same underlying CR-structure.

Sasaki metrics with special geometric properties

SE, CSC and Extremal

- $\mathcal{S} = (\xi, \eta, \Phi, g)$ is Sasaki-Einstein (SE) (meaning the Sasaki metric is Einstein) $\iff$ the cone metric on $C(M)$ above is Ricci-Flat.
- If $\mathcal{S} = (\xi, \eta, \Phi, g)$ is SE, then (g_T, ω_T) is positive Kähler Einstein.
- Any Sasaki structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ satisfying that (g_T, ω_T) is Kähler Einstein is called η-Einstein.
- $\mathcal{S} = (\xi, \eta, \Phi, g)$ is η-Einstein $\iff$ its entire ray is η-Einstein.
- If (g_T, ω_T) is positive Kähler Einstein there there is exactly one Sasaki-Einstein metric in the η-Einstein ray of $\mathcal{S} = (\xi, \eta, \Phi, g)$.

Sasaki metrics with special geometric properties

SE, CSC and Extremal

- $\mathcal{S} = (\xi, \eta, \Phi, g)$ has constant scalar curvature (CSC) $\iff$ the transverse Kähler structure has constant scalar curvature.
- $\mathcal{S} = (\xi, \eta, \Phi, g)$ has CSC iff its entire ray has CSC ("CSC ray").
- CSC can be generalized to Sasaki Extremal (Boyer-Galicki-Simanca) such that
- $\mathcal{S} = (\xi, \eta, \Phi, g)$ is extremal $\iff$ the transverse Kähler structure is extremal
- $\mathcal{S} = (\xi, \eta, \Phi, g)$ is extremal $\iff$ the entire ray is extremal.
- We will say that $\mathcal{S} = (\xi, \eta, \Phi, g)$
  is SE/η-Einstein/CSC/extremal whenever it is CSC/extremal up to isotopy.
- SE $\subset$ η-Einstein $\subset$ Sasaki CSC $\subset$ Sasaki Extremal

Yamazaki's Fiber Joins

Upshot

T. Yamazaki introduced the fiber join for K-contact structures. Here we will extend this to Sasaki structures.

Set-Up

- Suppose N is a Hodge (Kähler) manifold and for $i = 1, \ldots, d+1$, let ω_i be Kähler forms on N such that $[\omega_i] \in H^2(N, \mathbb{Z})$.
- Let L_i be a holomorphic line bundle over N such that $c_1(L_i) = [\omega_i]$. So $c_1(L_i)$ is in the Kähler cone of N.
- Each $c_1(L_i)$ also defines a principal S^1-bundle over N, $M_i \to N$, and we identify L_i with $M_i \times_{S^1} \mathbb{C}$.
- $M_i \xrightarrow{\pi} N$ has a natural Sasaki structure defined by the Boothby-Wang construction. Using $d\eta_i = \pi^*\omega_i$, the Sasaki structure on M_i is $[\xi_i, \eta_i, \Phi_i, g_i]$.

Yamazaki's Fiber Joins

The fiber join, $M_{\mathfrak{w}}$

Consider $\oplus_{j=1}^{d+1} L_j^*$ and equip each L_i^* with a Hermitian metric giving us a norm $r_i : L_i^* \to \mathbb{R}^{\geq 0}$ and polar coordinates (r_i, θ_i). Then the fiber join $M_{\mathfrak{w}}$ is the S^{2d+1}-bundle over N whose fibers are defined by $r_1^2 + \cdots + r_{d+1}^2 = 1$.

Now $M_{\mathfrak{w}}$ has a natural CR-structure $(\mathcal{D}, J)$ and a family, $\mathfrak{t}_{sph}^+(\mathcal{D}, J)$, of compatible Sasaki structures $\mathcal{S}_{\mathbf{w}} = (\xi_{\mathbf{w}}, \eta_{\mathbf{w}}, \Phi_{\mathbf{w}}, g_{\mathbf{w}})$, where $\xi_{\mathbf{w}} = \frac{1}{2}\sum_{j=1}^{d+1} w_j(\xi_j + \partial_{\theta_j})$ and $w_j \in \mathbb{R}^+$ are the *weights*. The Sasakian structure restricts to the standard weighted Sasakian structure on each fiber S^{2d+1}.
The transverse Kähler form is given by
$d\eta_{\mathbf{w}} = \sum_{j=1}^{d+1} \frac{1}{w_j}\left(r_j^2 d\eta_j + 2r_j dr_j \wedge (\eta_j + d\theta_j)\right)$
and the transverse Kähler metric is $g_{\mathbf{w}}^T = d\eta_{\mathbf{w}} \circ (\mathbb{1} \otimes \Phi_{\mathbf{w}})$.

Yamazaki's Fiber Joins

When $w_j \in \mathbb{Z}^+$, we have the commutative diagram of weighted S^1 actions:

$$\begin{array}{ccccc}
S^1_{\mathbf{w}} & \longrightarrow & S^{2d+1} & \longrightarrow & \mathbb{CP}^d[\mathbf{w}] \\
\downarrow{\scriptstyle id} & & \downarrow & & \downarrow \\
S^1_{\mathbf{w}} & \longrightarrow & M_{\mathfrak{w}} & \longrightarrow & \mathbb{P}_{\mathbf{w}}(\oplus_{j=1}^{d+1} L_j^*) \\
 & & \downarrow & & \downarrow \\
 & & N & \xrightarrow{id} & N.
\end{array} \tag{1}$$

Yamazaki's Fiber Joins

Theorem (From work in progress with C.P. Boyer)

For $d = 1$ and co-prime $\mathbf{w} = (w_1, w_2) \in (\mathbb{Z}^+)^2$, the quasi-regular quotient of $M_{\mathfrak{w}}$ with respect to $\xi_{\mathbf{w}}$ is the log pair $B_{\mathfrak{w},\mathbf{w}} := (\mathbb{P}((L_1^*)^{w_2} \oplus (L_2^*)^{w_1}), \Delta_{\mathbf{w}})$, where $\Delta_{\mathbf{w}} = (1 - 1/w_1)D_1 + (1 - 1/w_2)D_2$ and D_1, D_2 are the zero and infinity sections, respectively, of the bundle $\mathbb{P}((L_1^*)^{w_2} \oplus (L_2^*)^{w_1}) \to N$. Moreover, up to scale, the induced transverse Kähler class on $B_{\mathfrak{w},\mathbf{w}}$ is equal to $2\pi(w_2[\omega_1] + w_1[\omega_2]) + \Xi)$ where $c_1(L_j) = [\omega_j]$ and $\Xi/(2\pi)$ is the Poincare dual of $(D_1 + D_2)$.

Remarks:

- This kind of theorem can help us in explicitly exploring the existence of Sasaki metrics with special geometries on $M_{\mathfrak{w}}$.
- For general $d \geq 1$, we just know (for now) that the regular quotient is $(\mathbb{P}(L_1^* \oplus \cdots \oplus L_{d+1}^*) \to N$. This can still be useful.

Examples and Results for $d = 1$

The base $N = \Sigma_1 \times \Sigma_2$:

Let Σ_1 and Σ_2 be two compact Riemann surfaces of genus $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively. Let ω_{Σ_j} be the constant curvature, unit volume, Kähler form on Σ_j (such that the Ricci form satisfies $\rho_j = 2(\mathcal{G}_j - 1)\omega_{\Sigma_j}$).

Note

$Span_{\mathbb{R}^+}\{[\omega_{\Sigma_1}], [\omega_{\Sigma_2}]\}$ gives us Kähler classes on $N = \Sigma_1 \times \Sigma_2$, where we (here and onwards) think of ω_{Σ_i} as pulled back to N.

The line bundles:

Let $k_i^j \in \mathbb{Z}^+$ and consider the Kähler forms $\omega_i = k_i^1 \omega_{\Sigma_1} + k_i^2 \omega_{\Sigma_2}$ on N. Let L_i be a holomorphic line bundle over N such that $c_1(L_i) = [\omega_i]$, for $k_i^j \in \mathbb{Z}^+$. The overall choice of L_1 and L_2 can be given (up to multiplying by flat holomorphic line bundles) by the matrix $K = \begin{pmatrix} k_1^1 & k_1^2 \\ k_2^1 & k_2^2 \end{pmatrix}$.

Examples and Results for $d = 1$

Yamazaki Join

- For a given K we denote the $d = 1$ Yamazaki join by M_K.
- For $\mathcal{G}_i \geq 1$, $i = 1, 2$, $\mathfrak{t}^+_{sph}(\mathcal{D}, J)$, is the entire (unreduced) Sasaki cone $\mathfrak{t}^+$ of $(\mathcal{D}, J)$, but in general $\mathfrak{t}^+_{sph}(\mathcal{D}, J)$ is a proper subcone of $\mathfrak{t}^+$.
- The regular quotient of M_K with respect to ξ_1 is $\mathbb{P}(L_1^* \oplus L_2^*) \to N$ or $\mathbb{P}(\mathbb{1} \oplus (L_1 \otimes L_2^*)) \to N$ or $\mathbb{P}(\mathbb{1} \oplus \mathcal{O}(k_1^1 - k_2^1, k_1^2 - k_2^2)) \to N$. We recognize this quotient (in most cases) as a so-called admissible projective bundle (V. Apostolov, D. M. J. Calderbank, P. Gauduchon, T-F).
- The transverse Kähler class of the regular Sasaki structure is given (up to scale) by $(k_1^1 + k_2^1)[\omega_{\Sigma_1}] + (k_1^2 + k_2^2)[\omega_{\Sigma_2}] + \Xi$. We recognize this as an admissible Kähler class which we can work explicitly with.
- If $det\, K = 0$, then this case reduces to a so-called $S^3_{\mathbf{w}}$-join (due to M.Y. Wang-W. Ziller and C.P. Boyer-K. Galicki). This construction was used in earlier work with C.P. Boyer. This is the decomposable case.

Obstruction to Sasaki-Einstein metrics on M_K

Obstruction to Sasaki-Einstein metrics on M_K

$$\begin{aligned} & c_1(\mathcal{D}) \\ = \quad & \pi^*\left(-(k_1^1+k_2^1)[\omega_{\Sigma_1}]-(k_1^2+k_2^2)[\omega_{\Sigma_2}]+c_1(N)\right) \\ = \quad & \pi^*\left(-(k_1^1+k_2^1)[\omega_{\Sigma_1}]-(k_1^2+k_2^2)[\omega_{\Sigma_2}]+c_1(\Sigma_1\times\Sigma_2)\right) \\ = \quad & \pi^*\left(\left(2(1-\mathcal{G}_1)-(k_1^1+k_2^1)\right)[\omega_{\Sigma_1}]+\left(2(1-\mathcal{G}_2)-(k_1^2+k_2^2)\right)[\omega_{\Sigma_2}]\right) \end{aligned}$$

This vanishes if and only if $\mathcal{G}_i = 0$ and $k_j^i = 1$, $i = 1, 2$, $j = 1, 2$. Then M_K is just a S^1-bundle over $\mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1$.

Note

So the Koiso-Sakane Kähler-Einstein metric on $\mathbb{P}(\mathbb{1} \oplus \mathcal{O}(1,-1)) \to \mathbb{CP}^1 \times \mathbb{CP}^1$ does not appear as a regular quotient of any of our Yamazaki fiber joins.

Some Results for M_K

Theorem (From joint work with V. Apostolov and G. Maschler as well as with C. P. Boyer , E. Legendre, and H. Huang)

For $\mathcal{G}_1, \mathcal{G}_2 \leq 1$, the Sasaki cone of M_K has a 2-dimensional subcone that is saturated by Sasaki rays with extremal Sasaki metrics in their isotopy class. Moreover, this subcone contains a ray with constant scalar curvature (CSC) Sasaki metrics (up to isotopy).

Proof in bullet points.

- Use the admissible set-up for the regular quotient to parametrize (up to scale) the 2-dim Sasaki cone.
- Use the CR-twist method by V. Apostolov and D.M.J. Calderbank together with known existence results from joint work with V. Apostolov and G. Maschler to show that we have extremal Sasaki metrics (up to isotopy) for each ray in the Sasaki-cone.
- Calculate the (relevant part of the) Sasaki-Futaki invariant to show that one of those rays must be CSC.

□

Some Results for M_K

Example

Up to scale, $\mathbb{P}(\mathbb{1} \oplus \mathcal{O}(1,-1)) \to \mathbb{CP}^1 \times \mathbb{CP}^1$ has a two-parameter continuous family of CSC Kähler classes. There is a discrete one-parameter subfamily of those CSC Kähler classes that may appear as regular quotients of a Yamazaki join as above. The rest (including the KE class) do not show up as such.

Note

From joint work with C. P. Boyer , E. Legendre, and H. Huang we also know of non-existence results for certain combinations of $\mathcal{G}_1, \mathcal{G}_2 > 1$ and K.

Some Results for M_K

Theorem (From current work in progress with C.P. Boyer)

For all $\mathcal{G}_1, \mathcal{G}_2 \geq 1$, there exist matrices $K = \begin{pmatrix} k_1^1 & k_1^2 \\ k_2^1 & k_2^2 \end{pmatrix}$. such that the entire Sasaki cone of M_K is extremal and contains a CSC ray.

Proof in bullet points.

- Let K be **carefully chosen**! It needs to depend on $\mathcal{G}_1$ and $\mathcal{G}_2$.
- Use the admissible set-up for the regular quotient to parametrize (up to scale) the 2-dim Sasaki cone.
- Use the CR-twist method by V. Apostolov and D.M.J. Calderbank to show that for our carefully chosen K, we have extremal Sasaki admissible metrics (up to isotopy) for each ray in the Sasaki-cone.
- Calculate the (relevant part of the) Sasaki-Futaki invariant to show that one of those rays must be CSC.

□

Some Results for M_K

Variation:

We can show a similar type of result for the case where $N = \mathbb{P}(E) \to \Sigma_g$, where $E \to \Sigma_g$ is a indecomposable polystable rank 2 holomorphic vector bundle over a compact Riemann surface of genus $\mathcal{G} > 1$.

Here, the line bundles are given by
$c_1(L_j) = [\omega_j] == k_j^1 \mathbf{h} + (\frac{k_j^1}{2}(degE) + k_j^2)\mathbf{f}$, where $\mathbf{f} \in H^2(N, \mathbb{Z})$ denote the class of the fiber of $\mathbb{P}(E) \to \Sigma_g$ and $\mathbf{h} \in H^2(N, \mathbb{Z})$ denote the class of the (E-dependent) tautological line bundle on N. We need:

- If $degE$ is even, $k_j^i \in \mathbb{Z}^+$
- If $degE$ is odd, one of the following is true:
  - k_j^1 is an even positive integer and $k_j^2 \in \mathbb{Z}^+$
  - k_j^1 is an odd positive integer and $(k_j^2 - 1/2) \in \mathbb{Z}^+$.

Then a choice of $K = \begin{pmatrix} k_1^1 & k_1^2 \\ k_2^1 & k_2^2. \end{pmatrix}$ yields a Yamazaki fiber join $M_{\mathfrak{w}} = S(L_1^* \oplus L_2^*)$. Again, for the result, **choose K carefully**.

More general results for $d \geq 1$

Definition

For the fiber join $M_{\mathfrak{w}}$ we have in particular that the complex manifold arising as the quotient of the regular Reeb vector field ξ_1 is equal to $\mathbb{P}\left(\oplus_{j=1}^{d+1} L_j^*\right) \to N$. This is an admissible projective bundle, as defined in joint work with V. Apostolov, D.M.J. Calderbank, and P. Gauduchon, **iff**

1. The base N is a local product of Kähler manifolds (N_a, Ω_a), $a \in \mathcal{A} \subset \mathbb{N}$, where $\mathcal{A}$ is a finite index set.
2. There exist $d_0, d_\infty \in \mathbb{N} \cup \{0\}$, with $d = d_0 + d_\infty + 1$, such that $E_0 := \oplus_{j=1}^{d_0+1} L_j^*$ and $E_\infty := \oplus_{j=d_0+2}^{d_0+d_\infty+2} L_j^*$ are both projectively flat hermitian holomorphic vector bundles. This would, for example, be true if $L_j^* = L_0$ for $j = 1, ..., d_0+1$ and $L_j^* = L_\infty$ for $j = d_0+2, ..., d_0+d_\infty+2$, where L_0 and L_∞ are some holomorphic line bundles. That is, $E_0 = L_0 \otimes \mathbb{C}^{d_0+1}$ and $E_\infty = L_\infty \otimes \mathbb{C}^{d_\infty+1}$. More generally, $c_1(L_1^*) = \cdots = c_1(L_{d_0+1}^*)$ and $c_1(L_{d_0+2}^*) = \cdots = c_1(L_{d_0+d_\infty+2}^*)$ would be sufficient.

More general results for $d \geq 1$

Definition

3. $\frac{c_1(E_\infty)}{d_\infty+1} - \frac{c_1(E_0)}{d_0+1} = \sum_{a\in\mathcal{A}}[\epsilon_a\Omega_a]$, where $\epsilon_a = \pm 1$.

The Kähler cone of the total space of an admissible bundle $\mathbb{P}(E_0 \oplus E_\infty) \to N$ has a subcone of so-called admissible Kähler classes. Up to scale, they look as follows:

$$\sum_{a\in\mathcal{A}} \frac{1}{x_a}[\Omega_a] + \Xi,$$

where $0 < \epsilon_a x_a < 1$ and Ξ is a fixed class. This subcone has dimension $|\mathcal{A}| + 1$ and, in general, this is not the entire Kähler cone.

Any fiber join $M_{\mathfrak{w}}$ where the quotient of the regular Reeb vector field ξ_1 is admissible will also be called admissible. If further the transverse Kähler class of this quotient is admissible, then we call $M_{\mathfrak{w}}$ superadmissible.

More general results

Theorem (Follows from joint work with C.P. Boyer, E. Legendre, and H. Huang)

Let $M_{\mathfrak{w}}$ be a superadmissible fiber join whose regular quotient is a ruled manifold of the form $\mathbb{P}(E_0 \oplus E_\infty) \longrightarrow N$ where E_0, E_∞ are projectively flat hermitian holomorphic vector bundles on N of complex dimension $(d_0+1), (d_\infty+1)$, respectively, and N is a local Kähler product of non-negative CSC metrics. Then $\mathfrak{t}^+_{sph}$ has a 2-dimensional subcone of extremal Sasaki metrics which contains at least one ray of CSC Sasaki metrics

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More general results

Proof in bullet points.

- Use the admissible set-up for the regular quotient to parametrize (up to scale) a 2-dim sub-cone of $\mathfrak{t}^+_{sph}$.
- Use the CR-twist method by V. Apostolov and D.M.J. Calderbank together with known existence results from joint work with V. Apostolov and G. Maschler to show that we have extremal Sasaki metrics (up to isotopy) for each ray in the Sasaki-cone.
- Use the (relevant part of the) Sasaki-Futaki invariant and its interplay with an Einstein Hilbert functional to show that one of those rays must be CSC.

□

The CR-twist method by V. Apostolov and D.M.J. Calderbank

Set-Up

Let f be a positive Killing potential on a Kähler manifold (N, J, g, ω) of complex dimension m and assume that $[\omega/2\pi]$ is an integer cohomology class. Then f determines a certain Sasaki structure in the Sasaki-Reeb cone of the Boothby-Wang constructed Sasaki manifold over (M, g, ω).

Theorem Apostolov-Calderbank

This Sasaki structure determined by f is extremal if and only if g is $(f, m+2)$-extremal, i.e. if and only if

$$Scal_{f,m+2}(g) = f^2 Scal(g) - 2(m+1) f \Delta_g f - (m+2)(m+1)|df|_g^2$$

is a Killing potential.

The CR-twist method by V. Apostolov and D.M.J. Calderbank

Constant Scalar Curvature

Note that $\frac{Scal_{f,m+2}(g)}{f}$ is equal to the transverse scalar curvature of the Sasaki structure induced by f and so, **this Sasaki structure has constant scalar curvature if and only if** $\frac{Scal_{f,m+2}(g)}{f}$ **equals a constant.**

Mixing this with the superadmissible case

We can use the following existence criterion applied to the regular and admissible quotient.

The CR-twist method in the admissible case

Lemma

Let Ω be a rational admissible Kähler class on an admissible Kähler manifold $N^{ad} = \mathbb{P}(E_0 \oplus E_\infty) \to N$ where N is a compact Kähler manifold which is a local product of CSCK metrics. Pick any admissible Kähler metric g with Kähler form in Ω and let the moment map be given by $\mathfrak{z} : N^{ad} \to [-1, 1]$. Let $(M, \mathcal{S})$ be the Boothby-Wang constructed Sasaki manifold given by an appropriate rescale of Ω. Let $c \in (-1, 1)$ and consider the corresponding weighted extremal polynomial $F_{\Omega,c,m+2}(\mathfrak{z})$ given by setting up the ansatz for admissible $(f, m+2)$-extremal Kähler metrics (follows from joint work with V. Apostolov and G. Maschler). Then

1. (essentially V. Apostolov, D.M.J. Calderbank, and E. Legendre) the Reeb vector field K_c determined by $f_{K_c} = c\,\mathfrak{z} + 1$ is extremal (up to isotopy) if and only if

$$F_{\Omega,c,m+2}(\mathfrak{z}) > 0, \quad -1 < \mathfrak{z} < 1$$

The CR-twist method in the admissible case

Lemma-continued

- the Reeb vector field K_c determined by $f_{K_c} = c\,\mathfrak{z} + 1$ is CSC (up to isotopy) if and only if both of the following hold:
  1. $$F_{\Omega,c,m+2}(\mathfrak{z}) > 0, \quad -1 < \mathfrak{z} < 1$$
  2. (vanishing of the (relevant part of the) Futaki-Invariant)
  $$\alpha_{1,-m-2}\beta_{0,(-m-1)} - \alpha_{0,-m-2}\beta_{1,(-m-1)} = 0,$$
  where $\alpha_{r,k}$ and $\beta_{r,k}$ are values determined by the particular admissible data, which in turn is determined by the particular superadmissible Yamazaki join.

The CR-twist method in the admissible case

Remarks:

- The $(d = 1)$ case where $N = \Sigma_{\mathcal{G}_1} \times \Sigma_{\mathcal{G}_2}$ and $\mathcal{G}_1, \mathcal{G}_2 \leq 1$ is a pretty straight forward.
- In the $(d = 1)$ case where $N = \Sigma_{\mathcal{G}_1} \times \Sigma_{\mathcal{G}_2}$ and $\mathcal{G}_i > 1$ for at least one of $i = 1, 2$, property (1) is not guaranteed and care must be taken in choosing K. Property (2) is as straightforward as above.
- In the $d \geq 1$ superadmissible case (with N being a local Kähler product of non-negative CSC metrics), (1) follows from existence results in joint work with V. Apostolov and G. Maschler (which in turn is an adaptation of the root counting argument due to works by A. Hwang and D. Guan.

  We needed a delicate argument for (2). This is where the Einstein Hilbert functional became really useful.

Thank you!