  

Hard Lefschetz Property for Isometric Flows and $\mathbb{S}^3$-actions

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Sasakian manifolds, Riemanniann foliations, and related topics.
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Sketch of the talk

1. HLP for isometric flows
2. Isometric Flows
3. Our results
4. HLP for $\mathbb{S}^3$-actions

Symplectic Manifolds

- M closed C^∞ manifold.
- $\dim M = 2n$.
- We work with de Rham cohomology
- $\omega \in \Omega^2(M)$
- $d\omega = 0$ and $\omega^n \in \Omega^{2n}(M)$ volume form $\Longrightarrow [\omega^n]$ generates H^{2n}_M
- $H^0_M = \langle 1 \rangle$
- $H^{2n}_M = \langle [\omega]^n \rangle$

Hard Lefschetz Property (HLP)

- (M, ω) satisfies HLP if

$$\begin{array}{rcl} L^{n-k} \colon H^k_M & \longrightarrow & H^{2n-k}_M \\ [\beta] & \longmapsto & [\beta \wedge \omega^{n-k}] \end{array}$$

  are isomorphisms $\forall k = 0, \ldots, n$.

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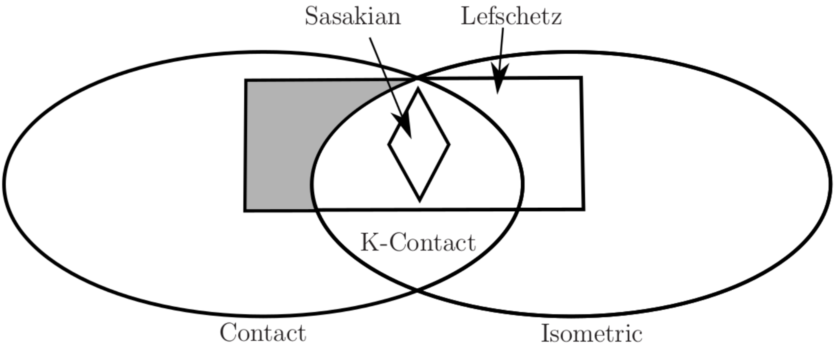
$$H^0_M \quad H^1_M \quad \ldots H^{2n} \ldots \quad H^{2n-1}_M \quad H^{2n}_M$$

$$L^n = \wedge[\omega]^n L^{n-1} = \wedge[\omega]^{n-1} L^0 = id_M$$

Contact manifolds (flows)

- $\dim M = 2n + 1$
- $\eta \in \Omega^1(M)$ contact form
- $\eta \wedge (d\eta)^n$ volume form of M.
- $\xi \in \mathfrak{X}(M)$ Reeb vector field
- $\eta(\xi) = 1$ and $L_\xi \eta = 0$.

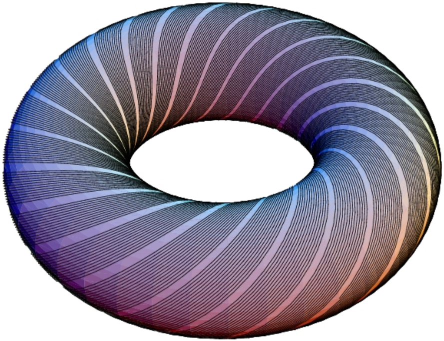
HLP for flows on M^{2n+1}



- [Capelleti-Montano et al., JDG 2015] Sasakian HL_c
- [Lin Yi, arXiv 2016] K-contact $THL \Longleftrightarrow HL$
- [RSW, TG 2022] Isometric $THL \Longleftrightarrow HL$

Flows

- M closed manifold (of dimension M^{2n+1})
- $\mathscr{F}$ is a flow (1 dimensional oriented foliation) over $M \Leftrightarrow X$ vector field $\Leftrightarrow \Phi$ $\mathbb{R}$-action
- ω basic if $i_X\omega = i_x d\omega = 0$.
- Basic cohomology H_B^*: "cohomology of the quotient space";



- **Example:** linear flows on tori.
- **Example:** linear flows on odd spheres.

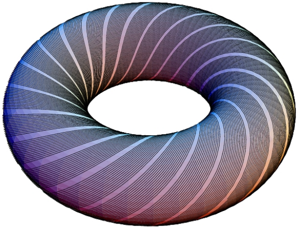
Isometric Flows

An almost free action

$$\Phi\colon \mathbb{R} \times (M, \mu) \longrightarrow (M, \mu)$$

preserving a Riemannian metric μ on M.

Example:



- $M = \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}$
- Action $\Phi(t, [x, y]) = [at + x, bt + y]$ $(a^2 + b^2 = 1)$
- Equivalently, $X = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}$

Gysin sequence for isometric flows

- $\chi = i_X\mu \in \Omega^1(M)$ characteristic form of the flow. $L_X\chi = 0$
- Euler form: $d\chi \in \Omega^2(M/\mathscr{F})$ basic form
- We have

$$\Omega^*(M/\mathscr{F}) \oplus \Omega^{*-1}(M/\mathscr{F}) \cong \Omega^*(M)^{\mathbb{R}}$$

$(\alpha, \beta) \mapsto \alpha + \chi \wedge \beta$

- which yields the (exact) Gysin sequence

$$\cdots \to H_B^i \to H_M^i \xrightarrow{\int = i_X} H_B^{i-1} \xrightarrow{\wedge [e]} H_B^{i+1} \to \dots$$

- The connecting morphism is multiplication by the Euler class

$$[e] \in H_B^2.$$

Transverse Hard Lefschetz (THL) property for isometric flows

$\mathscr{F}$ satisfies the *Transversal Hard Lefschetz property* if the following maps are isomorphisms for $k = 0, \ldots, n$:

$$L^{n-k} \colon H_B^k \longrightarrow H_B^{2n-k},$$

where

$$L^{n-k}([\beta]) = [\beta \wedge e^{n-k}].$$

(Global) Hard Lefschetz (HL) property for isometric flows [RSW]

- $\mathscr{F}$ satisfies the *(global) Hard Lefschetz property* if there exist isomorphisms

$$\mathscr{L}^{n-k} \colon H_M^k \longrightarrow H_M^{2n-k+1}$$

making the following diagram commutative:

$$\begin{array}{ccccc}
H_M^k & & \xrightarrow{\quad \mathscr{L}^{n-k} \quad} & & H_M^{2n-k+1} \\
\uparrow{\scriptstyle i_k} & & & & \downarrow{\scriptstyle \int} \\
PH_B^k & \hookrightarrow & H_B^k & \xrightarrow{L^{n-k}} & H_B^{2n-k}
\end{array}$$

for $k = 0, \ldots, n$.

- Basic primitive cohomology group:

$$PH_B^k = \left\{ [\beta] \in H_B^k \;\middle|\; [\beta \wedge e^{n-k+1}] = 0 \text{ in } H_B^{2n-k+2} \right\}.$$

Main Result

Theorem (RSW, Transformation Groups 2022)

Let $\mathscr{F}$ be an isometric flow on a closed manifold of odd dimension. Then,

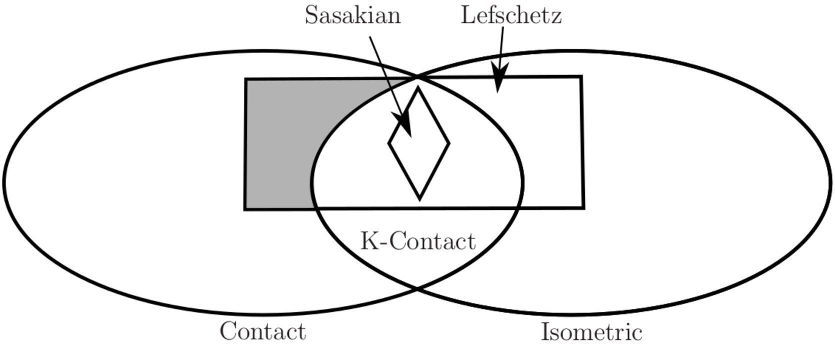
$$THL \Longleftrightarrow HL$$

Definition

An isometric flow on a closed manifold of odd dimension is said to be a *Lefschetz isometric flow* if it satisfies HL or, equivalently, THL.

- Being Lefschetz does NOT depend on the metric (Euler classes are proportional).

HL_c for contact manifolds [Cappeletti-Montano et.al,JDG 2015]

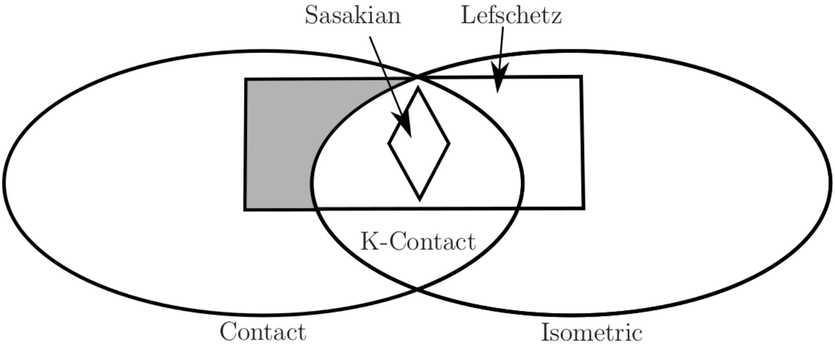


A contact manifold (M, η) with Reeb vector field ξ is a *Lefschetz contact manifold* if for every $k \leq n$, the relation between H^k_M and H^{2n+1-k}_M defined by

$$\mathcal{R}^k = \{ ([\beta], [\eta \wedge (d\eta)^{n-k}\beta]) \mid \beta \in \Omega^k_M, d\beta = 0, i_\xi\beta = 0, (d\eta)^{n-k+1} \wedge \beta = 0 \}$$

is the graph of an isomorphism $H^k_M \cong H^{2n-k+1}_M$.

$HL_c \Longleftrightarrow THL$ for K-contact [Lin Yi, arXiv 2016]

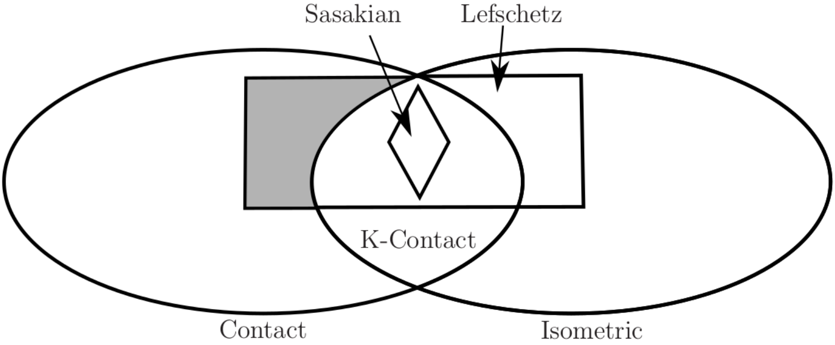


Theorem (Lin Yi, arXiv 2016)

For any K-contact flow on a closed manifold,

$$HL_c \Longleftrightarrow THL_c$$

Generalization



Proposition (RSW, 2022)

The isometric flow defined by the Reeb vector field of a K-contact manifold is an isometric Lefschetz flow if and only if it satisfies HL_c

Example of a Lefschetz non-contact isometric flow [RSW, TG 2022]

- Let $B = \mathbb{CP}^2 \sharp \mathbb{CP}^2$. Recall that its cohomology is given by:

$$\begin{array}{ll} H^0_B = \mathbb{R} & H^2_B =< [a], [b] >= \mathbb{R} \oplus \mathbb{R} \\ H^1_B = H^3_B = 0 & H^4_B =< [a]^2 >=< [b]^2 >= \mathbb{R}, \end{array}$$

- Take $e = a$ and apply Kobayashi (a.k.a. Boothby-Wang).
- $\dim(M) = 5$ with $\mathbb{S}^1$-action and $[e] = [a] \in H^2_B$.
- B not symplectic (M.Audin)$\Longrightarrow$ M not contact.
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$$L^2 \colon H^0(B) \xrightarrow{\wedge [a]^2} H^4(B) \qquad L^1 \colon H^1(B) \xrightarrow{\wedge [a]} H^3(B)$$

$$L^0 \colon H^2(B) \xrightarrow{Id} H^2(B)$$

give THL.

Almost free actions of $\mathbb{S}^3$

- M closed manifold of dimension $4n+3$
- Φ smooth almost free $\mathbb{S}^3$-action on M.
- $B = M/\mathbb{S}^3$ is an orbifold of dimension $4n$.
- orbits define a foliation $\mathscr{F}_{\mathbb{S}^3}$ of dimension 3.
- Basic cohomology $H_B^* = H^*(M/\mathscr{F}_{\mathbb{S}^3})$.

Actors of the Gysin sequence

- $X_i,\ i = 1, 2, 3$ the three fundamental vector fields of the $\mathbb{S}^3$-action.
- $\mathbb{S}^3$-invariant metric μ.
- dual characteristic forms are $\chi_i = i_{X_i}\mu$.
-
$$\begin{aligned} d\chi_1 &= e_1 - \chi_2 \wedge \chi_3 \\ d\chi_2 &= e_2 + \chi_1 \wedge \chi_3 \\ d\chi_3 &= e_3 - \chi_1 \wedge \chi_2 \end{aligned}$$
- e_i is $\mathscr{F}_i$-basic (the Euler form of the $\mathbb{S}^1$-action-flow $\mathscr{F}_i$).
- $e_i \in \Omega^2(M)$ are $\mathbb{S}^3$-horizontal forms, but not $\mathscr{F}_{\mathbb{S}^3}$-basic.
-
$$d(\overbrace{e_1\chi_1 + e_2\chi_2 + e_3\chi_3}^{\theta}) = \overbrace{e_1^2 + e_2^2 + e_3^2}^{\Omega} + d(\chi_1\chi_2\chi_3)$$
- The *Euler form* of $\Omega = e_1^2 + e_2^2 + e_3^2$ is $\mathscr{F}_{\mathbb{S}^3}$-basic.
- $[\Omega] \in H_B^4$ is the *Euler class* of Φ.

Gysin sequence of Φ

- The classical Gysin sequence for free $\mathbb{S}^3$-actions works if the action is almost-free.
- $$\cdots \to H_B^i \xrightarrow{\iota} H_M^k \xrightarrow{\int} H_B^{k-3} \xrightarrow{L} H_B^{k+1} \xrightarrow{\iota} H_M^{k+1} \to \dots$$
- ι is induced by the inclusion
- $\int$ is the operator $i_{X_1} i_{X_2} i_{X_3}$
- the connecting morphism L is the multiplication by the Euler class $[\Omega]$.

Hard Lefschetz Property (THL) for Φ

- $L\colon H_B^* \to H_B^{*+4}$ multiplication by the Euler class.
- We say that Φ safisfies *THL* if

$$L^{n-k}\colon H_B^{2k} \longrightarrow H_B^{4n-2k}$$

defined as $L^{n-k}([\alpha]) = [\alpha \wedge \Omega^{n-k}]$, is an isomorphism for $k = 0, \ldots, n$.

- $$H_B^0 \quad H_B^1 \quad H_B^1 \quad \ldots H^{2n}{}_B \ldots \quad H_B^{4n-2} \quad H_B^{4n-1} \quad H_B^{4n}$$

$$L^n = \wedge[\Omega]^n L^{n-1} = \wedge[\Omega]^{n-1} L^0 = id$$

(global) Hard Lefschetz Property (HL) for Φ

- The Primitive basic cohomology group of dimension $2k$ is

$$PH_B^{2k} = \{ [\beta] \in H_B^{2k} \mid [\beta \wedge \Omega^{n-k+1}] = 0 \}$$

- Φ safisfies *HL* if there exist isomorphisms $\mathscr{L}^{n-k} \colon H_M^{2k} \longrightarrow H_M^{4n-2k+3}$ such that the following diagram is commutative for $k = 0, \ldots, n$:

$$\begin{array}{ccccc} & & H_M^{2k} & \xrightarrow{\mathscr{L}^{n-k}} & H_M^{4n-2k+3} \\ & \overset{i_k}{\nearrow} & & & \downarrow \int \\ PH_B^{2k} & \hookrightarrow & H_B^{2k} & \xrightarrow{L^{n-k}} & H_B^{4n-2k} \end{array} ,$$

where i_k is induced by the inclusion of forms.

Equivalence of THL and HL for Φ

- Φ satisfies the *injectivity condition* if the multiplication by the Euler class induces a monomorphism: $L\colon H_B^{2k-3} \rightarrowtail H_B^{2k+1}$ for $k = 0, \ldots, n$.

Theorem (RSW)

Let Φ be an almost free smooth action of $\mathbb{S}^3$ on a closed manifold of dimension $4n+3$ satisfying the injectivity condition. Then, it satisfies HL if and only if it satisfies THL.

3-Sasakian manifolds

- Consider the almost free action of $\mathbb{S}^3$ induced by a 3-Sasakian structure.
- The small basic Betti numbers are zero:

$$b_B^{2k+1} = 0, \ \forall k = 0, 1, \ldots, n$$

$\Longrightarrow$ injectivity condition.

Corollary (RSW)

3-Sasakians satisfy THL and HL.

Example of a Lefschetz $\mathbb{S}^3$-action not 3-Sasakian [RSW]

- There are infinitely many distinct free smooth actions of $\mathbb{S}^3$ on $M = \mathbb{S}^{11}$ ([Hsiang, Q.J.M.Ox., 1964])
- There are only a finite amount of 3-Sasakian structures on $\mathbb{S}^{11}$ [Boyer,Galicki]
- So, take Φ an action that is not 3-Sasakian (there are infinitely many!)
- Gysin sequence $\Longrightarrow$ injectivity condition for all free actions on $\mathbb{S}^{11}$.
- So, Φ is Lefschetz (both HL and THL), but not 3-Sasakian.

J. I. Royo,M. Saralegi-Aranguren,R. Wolak, *Hard Lefschetz Property for Isometric Flows*, Transformation Groups, 2022
`www.ehu.eus/joseroyo`