A trace formula for foliated flows

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Sasakian manifolds, Riemannian foliations, and related topics
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Contents

Foliated flows

- M a closed manifold. $\mathcal{F}$ a foliation of codimension 1 on M.
- $\phi = \{\phi^t\}$ a foliated flow on M: ϕ^t : leaves $\to$ leaves.
- Preserved leaves: $\phi^t : L \circlearrowleft$
- $M^0 = \bigcup(\text{preserved leaves})$, saturated, closed.
- $M^1 = M \setminus M^0$, saturated, open, $\mathcal{F}^1 = \mathcal{F}|_{M^1}$.
- Σ: a complete transversal of $\mathcal{F}$.
- $\mathcal{H}$: holonomy pseudogroup of $\mathcal{F}$ on Σ (transverse dynamics of $\mathcal{F}$).
- ϕ foliated $\rightsquigarrow$ local flow $\bar{\phi} = \{\bar{\phi}^t\}$ on Σ, $\mathcal{H}$-equivariant.
- $\{\text{preserved leaves}\} \leftrightarrow \{\mathcal{H}\text{-orbits of points fixed by } \bar{\phi}\}$, $L \mapsto L \cap \Sigma$.

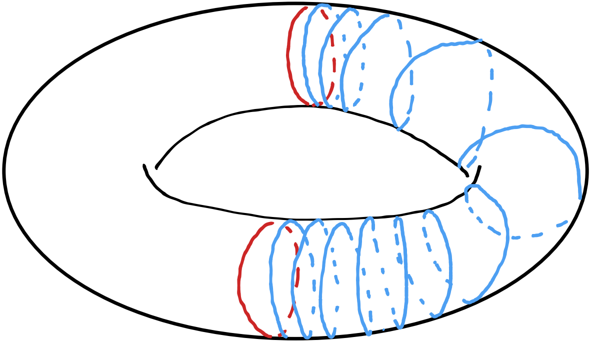
Simple foliated flows

The foliated flow ϕ is called simple if:

1. The fixed points $p \in \Sigma$ of $\bar{\phi}$ are simple:
   - $\mathrm{id} - \bar{\phi}_*^t : T_p\Sigma \equiv \mathbb{R} \circlearrowleft^{\cong}$ for $|t| \neq 0$ small.
   - $\rightsquigarrow \ \exists \varkappa \neq 0$ s.t. $\bar{\phi}_*^t \equiv e^{\varkappa t} : T_p\Sigma \equiv \mathbb{R} \circlearrowleft$.
   - $\mathcal{H}$-equivariance of $\phi \ \rightsquigarrow \varkappa = \varkappa_L$ where $L = L_p \subset M^0$.
   - $\rightsquigarrow \ M^0 = \bigcup_{\text{finite}} (\text{compact leaves})$.
2. $\phi \pitchfork \mathcal{F}$ on M^1.
3. The closed orbits $c \subset M^1$ are simple:
   - $\forall$ period $\ell \neq 0$, $\forall x \in c$, $\mathrm{id} - \phi_*^\ell : N_x c \equiv T_x\mathcal{F} \circlearrowleft^{\cong}$.
   - $\rightsquigarrow \epsilon_\ell(c) = \operatorname{sign}\det(\mathrm{id} - \phi_*^\ell : T_x\mathcal{F} \circlearrowleft)$.
   - $\ell(c) = \min\{\text{positive periods of } c\}$, other periods: $k\ell(c)$ $(k \in \mathbb{Z})$.

The foliations with simple foliated flows can be described . . .
(Kim-Masanori-Noda-Terashima 2020, Á.-Kordyukov-Leichtnam 2022).

A simple example



Reduced leafwise cohomology

- Exterior bundles over M: $\Lambda M = \bigwedge T^*M$, $\Lambda\mathcal{F} = \bigwedge T^*\mathcal{F}$.
- $C^\infty(M;\Lambda\mathcal{F}) = \{\text{leafwise forms}\}$, with the C^∞ topology,
  a topological vector space;
  a locally convex space (LCS);
  in fact, a Fréchet space.
- The leafwise differential operator $d_\mathcal{F} : C^\infty(M;\Lambda\mathcal{F}) \circlearrowleft$ is continuous,
  $\rightsquigarrow$ $(C^\infty(M;\Lambda\mathcal{F}), d_\mathcal{F})$ is a topological complex.
- $\rightsquigarrow$ leafwise cohomology: $H^\bullet(\mathcal{F})$ a possibly non-Hausdorff LCS
  (the differential complex $d_\mathcal{F}$ is not elliptic, only leafwise elliptic).
- Leafwise reduced cohomology: $\bar{H}^\bullet(\mathcal{F}) = H^\bullet(\mathcal{F})/\bar{0}$ Hausdorff LCS.
- $\rightsquigarrow$ $\phi^* = \{\phi^{t*}\}$ on $H^\bullet(\mathcal{F})$ and $\bar{H}^\bullet(\mathcal{F})$.
- It is invariant by leafwise homotopy equivalences of foliated flows.

Problem of the trace formula for simple foliated flows

Problem (C. Deninger 2001, Guillemin-Sternberg 1977 (partly))

- *Define a "Leftschetz distribution"* $L_{\mathrm{dis}}(\phi^t)$ *on* $\mathbb{R}$ *for* ϕ^{t*} *on* $\bar{H}^\bullet(\mathcal{F})$. $L_{\mathrm{dis}}(\phi^t) \in C^{-\infty}(\mathbb{R})$.
- *Prove a trace formula describing* $L_{\mathrm{dis}}(\phi^t)$ *in terms of the data from the closed orbits and preserved leaves.*
- *It should be similar to Weil's explicit formula in Arithmetics.*

Condition of transverse orientability

- Assume $\mathcal{F}$ is transversely oriented (for simplicity, not needed!):
  $N\mathcal{F} = TM/T\mathcal{F}$ is oriented, $\Leftrightarrow N\mathcal{F}$ is trivial.
- $\Leftrightarrow \exists \omega \in C^\infty(M, \Lambda^1 M)$ s.t. $T\mathcal{F} = \ker \omega$.
- $\exists \eta \in C^\infty(M; \Lambda^1 M)$ s.t. $d\omega = \eta \wedge \omega$ (integrability of $T\mathcal{F}$).
- $\eta \equiv \eta|_{\mathcal{F}} \in C^\infty(M; \Lambda\mathcal{F})$, $d_{\mathcal{F}}\eta = 0$, $\rightsquigarrow \xi = [\eta] \in H^1(\mathcal{F})$.
- ξ is determined by $\mathcal{F}$, but we can choose any $\eta \in \xi$.
- $\rightsquigarrow$ the normal bundle NM^0 is also trivial.
- $\rightsquigarrow$ a trivial tubular neighborhood of M^0: $T \equiv M^0 \times (-\epsilon, \epsilon)_\rho$.
- We can choose η and ρ such that $d_{\mathcal{F}}\rho = \rho\eta$.
- $\phi \pitchfork \mathcal{F}^1 \rightsquigarrow \mathcal{F}^1$ is always transversely oriented by ϕ.

Case without preserved leaves

- Assume $\nexists$ preserved leaves $(M^0 = \emptyset)$.
- $\rightsquigarrow \mathcal{F}$ is Riemannian: it is locally defined by Riemannian submersions for some Riemannian metric on M (a bundle-like metric).
- $\Leftrightarrow d\omega = 0 \quad (\eta = 0, \quad \omega$ is an invariant transverse volume form).
- $\rightsquigarrow \delta_{\mathcal{F}} = d^*_{\mathcal{F}}, \quad \Delta_{\mathcal{F}} = d_{\mathcal{F}}\delta_{\mathcal{F}} + \delta_{\mathcal{F}}d_{\mathcal{F}}$ on $C^\infty(M;\Lambda\mathcal{F})$, in $L^2(M;\Lambda\mathcal{F})$, (leafwise coderivative/Laplacian).
- $\Delta_{\mathcal{F}}$ is self-adjoint in $L^2(M;\Lambda\mathcal{F})$.
- $\rightsquigarrow$ leafwise heat operator $e^{-t\Delta_{\mathcal{F}}}$ on $L^2(M;\Lambda\mathcal{F}) \quad (0 < t \leq \infty)$
  $\Pi_{\mathcal{F}} = e^{-\infty\Delta_{\mathcal{F}}}$ is the orthogonal projection to $\ker(\Delta_{\mathcal{F}}$ in $L^2(M;\Lambda\mathcal{F}))$.

Case without preserved leaves

Theorem (Á.-Kordyukov 2001)

$(\alpha, t) \mapsto e^{-t\Delta_{\mathcal{F}}}\alpha$ *defines a smooth map*

$$C^\infty(M;\Lambda\mathcal{F}) \times [0,\infty] \to C^\infty(M;\Lambda\mathcal{F}) \,.$$

$\rightsquigarrow$ $\bar{H}^\bullet(\mathcal{F}) \cong \ker(\Delta_{\mathcal{F}}$ *in* $C^\infty(M;\Lambda\mathcal{F}))$.

Case without preserved leaves

- For $f \in C_c^\infty(\mathbb{R})$, let

$$P_f = \int_{\mathbb{R}} \phi^{t*} \cdot f(t)\, dt \circ \Pi_{\mathcal{F}} \quad \text{on} \quad C^\infty(M; \Lambda\mathcal{F})\ .$$

- $\omega \rightsquigarrow$ Connes' Euler characteristic (Connes 1979):

$$\chi_\omega(\mathcal{F}) = \int_M (\text{leafwise Euler density}) \cdot \omega\ .$$

Theorem (Á.-Kordyukov 2002)

1. *P_f is a smoothing operator, $\rightsquigarrow$ it is of trace class.*
2. $L_{\mathrm{dis}}(\phi) := (f \mapsto \operatorname{Str} P_f) \in C^{-\infty}(\mathbb{R})$.
3. $L_{\mathrm{dis}}(\phi) = \chi_\omega(\mathcal{F})\, \delta_0 + \sum_c \ell(c) \sum_{k \neq 0} \epsilon_c(k\ell(c))\, \delta_{k\ell(c)}\ .$

Idea of the proof

- $P_{u,f} = \int_{\mathbb{R}} \phi^{t*} \cdot f(t)\, dt \circ e^{-u\Delta_{\mathcal{F}}} \quad \forall u \in (0,\infty]$.
- $P_{u,f} : C^{-\infty}(M;\Lambda\mathcal{F}) \to C^{\infty}(M;\Lambda\mathcal{F})$ is continuous (smoothing).
- $\rightsquigarrow$ its Schwartz kernel $K_{u,f}(p,q)$ is C^∞ on $M \times M$,
- $(P_{u,f}\alpha)(x) = \int_M K_{u,f}(p,q)\alpha(y)$, $\operatorname{Str} P_{u,f} = \int_M \operatorname{str} K_{u,f}(p,p)$.
- $\lim_{u\downarrow 0} \operatorname{Str} P_{u,f} = \chi_\omega(\mathcal{F})\, f(0) + \sum_c \ell(c) \sum_{k\neq 0} \epsilon_c(k\ell(c))\, f(k\ell(c))$.
- $\operatorname{Tr}[A,B] = 0 \rightsquigarrow \frac{d}{du} \operatorname{Str} P_{u,f} = 0$.
- $\rightsquigarrow \operatorname{Str} P_f = \lim_{u\uparrow\infty} \operatorname{Str} P_{u,f} = \lim_{u\downarrow 0} \operatorname{Str} P_{u,f} =$ the above formula. $\square$

Another leafwise complex is needed

- Suppose $\exists$ preserved leaves $(M^0 \neq \emptyset)$.
- For simplicity, assume for a while $M^0 = L$, only one preserved leaf.
- $\rightsquigarrow (C^\infty(M; \Lambda\mathcal{F}), d_\mathcal{F})$ does not work!
- The dual complex $(C^{-\infty}(M; \Lambda\mathcal{F}), d_\mathcal{F})$ of leafwise currents does not work either.
- $\rightsquigarrow$ another leafwise complex is needed.
- $\rightsquigarrow$ deeper Functional Analysis.
- $\rightsquigarrow L_{\mathrm{dis}}(\phi)$ will be given by the action ϕ^* on 2 leafwise reduced cohomologies at the same time, $\bar{H}^\bullet I(\mathcal{F})$ and $\bar{H}^\bullet I'(\mathcal{F})$.

Distributions

- ΩM: the density bundle.
- $C^{-\infty}(M) = C^{\infty}(M; \Omega M)' = \{\text{distributions}\}$.
  (We consider continuous dual spaces with the strong topology.)
- $C^{\infty}(M) \subset C^{-\infty}(M)$ continuous, dense.
  (All inclusions of LCS considered here are continuous.)
- With more generality, we can consider spaces of special distributional sections of a vector bundle $E \to M$, like $C^{-\infty}(M; E)$. But their study can be reduced to corresponding spaces of distributions; e.g.,

$$C^{-\infty}(M; E) \equiv C^{-\infty}(M) \otimes_{C^{\infty}(M)} C^{\infty}(M; E) \; .$$

- $\rightsquigarrow$ for simplicity, we first study spaces of distributions.

Sobolev spaces

- Diff(M): differential operators on $C^\infty(M)$, also on $C^{-\infty}(M)$.
- Sobolev space of order $s \in \mathbb{N}_0$:

$$H^s(M) = \{\, u \in C^{-\infty}(M) \mid \mathrm{Diff}^s(M) \cdot u \subset L^2(M) \,\} ,$$

  with the coarser topology s.t. $P : H^s \to L^2$ is cont. $\forall P \in \mathrm{Diff}^s(M)$.
  (We always consider this kind of topology for spaces defined like this.)
- For $s \in \mathbb{R}$, the definition of $H^s(M)$ is similar but more involved ...
- $s < s' \Rightarrow H^s(M) \supset H^{s'}(M)$ dense.
- $\bigcup_s H^s(M) = C^{-\infty}(M)$, $\bigcap_s H^s(M) = C^\infty(M)$.
  (All unions/intersections of inductive systems of inclusions of LCS are endowed with the inductive/projective locally convex topologies.)
- If M is not compact, then $H^s(M)$ depends on the choice of a metric, $\rightsquigarrow$ unions/intersections $H^{\pm\infty}(M) \subset C^{\pm\infty}(M)$.

Conormal distributions

- Distributions with nice singularities at $M^0 = L$.
- $\mathfrak{X}(M, L) = \{ X \in \mathfrak{X}(M) \mid X \text{ tangent to } L \}$.
- $\rightsquigarrow \mathrm{Diff}(M, L) \subset \mathrm{Diff}(M)$.
- Distributions conormal at L of Sobolev order s:

$$I^{(s)}(M, L) = \{ u \in H^s(M) \mid \mathrm{Diff}(M, L) \cdot u \subset H^s(M) \} ,$$

  with the topology defined as above ...
- $s < s' \Rightarrow I^{(s)}(M, L) \supset I^{(s')}(M, L)$.
- Distributions conormal at L: $I(M, L) = \bigcup_s I^{(s)}(M, L)$.
- $C^{-\infty}(M) \supset I(M, L) \supset C^{\infty}(M)$ dense.

Conormal distributions (contd.)

Theorem (Á.-Kordyukov-Leichtnam 2023)

$I(M, L)$ is barreled, ultrabornological, webbed, acyclic Montel space,
$\rightsquigarrow$ complete, boundedly retractive and reflexive. (*It is a decent LCS.*)

- The proof uses Homological Theory of LCSs.

Dual-conormal distributions

- $I'(M,L) = I(M,L;\Omega M)' = \{$dual-conormal distributions$\}$.
- $C^{-\infty}(M) \supset I'(M,L) \supset C^{\infty}(M)$.
- $I'^{(s)}(M,L) = I^{(-s)}(M,L;\Omega M)'$.
- $s < s' \Rightarrow I'^{(s)}(M,L) \leftarrow I'^{(s')}(M,L)$ (restriction maps).

Theorem (Á.-Kordyukov-Leichtnam 2023)

1. $I'(M,L)$ *is a complete Montel space.*
2. $I'(M,L) = \varprojlim I'^{(s)}(M,L)$ *as* $s \downarrow -\infty$.
3. $I(M,L) \cap I'(M,L) = C^{\infty}(M)$.

Conormal distributions supported in L

- $K(M,L) = \{\, u \in I(M,L) \mid \operatorname{supp} u \subset L \,\} \subset I(M,L)$
  with the subspace topology.
- $K^{(s)}(M,L) \subset I^{(s)}(M,L)$ similar.

Theorem (Á.-Kordyukov-Leichtnam 2023)

1. $K(M,L)$ *satisfies the same properties as* $I(M,L)$.
2. $K(M,L) = \bigcup_s K^{(s)}(M,L)$.

Description of $K(M, L)$

- $C^\infty(L;\Omega^{-1}NL) \subset K(M,L)$:

$$\langle u, v\rangle = \int_L uv\ , \quad u \in C^\infty(L;\Omega^{-1}NL),\ v \in C^\infty(M;\Omega M)\ .$$

- $K(M,L) \supset \partial_\rho^m C^\infty(L;\Omega^{-1}NL) \overset{\partial_\rho^m}{\cong} C^\infty(L;\Omega^{-1}NL) \overset{\omega}{\cong} C^\infty(L).$

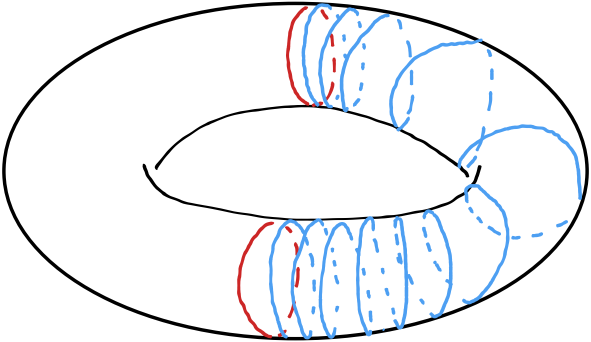
Theorem (Á.-Kordyukov-Leichtnam 2023)

$$K(M,L) = \bigoplus_{m=0}^{\infty} \partial_\rho^m C^\infty(L;\Omega^{-1}N) \cong \bigoplus_{m=0}^{\infty} C^\infty(L)\ .$$

Cutting M along L

- Cut M along L, $\rightsquigarrow \mathbf{M}$, $\boldsymbol{\mathcal{F}}$, $\partial\mathbf{M} \equiv L \sqcup L$, $\mathring{\mathbf{M}} \equiv M^1$, $\mathring{\boldsymbol{\mathcal{F}}} = \boldsymbol{\mathcal{F}}|_{\mathring{\mathbf{M}}} \equiv \mathcal{F}^1$, $\boldsymbol{\omega}$, $\boldsymbol{\eta}$, $\mathbf{T} \equiv \partial\mathbf{M} \times [0,\epsilon)_{\boldsymbol{\rho}} \equiv (L \times (-\epsilon, 0]_\rho) \sqcup (L \times [0,\epsilon)_\rho)$.
- $\exists$ an extension $\boldsymbol{\rho} \in C^\infty(\mathbf{M})$ s.t.:
  - $\boldsymbol{\rho} \geq 0$, $\partial\mathbf{M} = \{\boldsymbol{\rho} = 0\}$, $d\boldsymbol{\rho} \neq 0$ on $\partial\mathbf{M}$ (boundary defining function),
  - $\boldsymbol{\rho} \equiv |\rho|$ on $\mathbf{T}$,
  - $d_{\boldsymbol{\mathcal{F}}}\boldsymbol{\rho} = \boldsymbol{\rho}\boldsymbol{\eta}$.
- $\mathring{\boldsymbol{\mathcal{F}}} \equiv \mathcal{F}^1$ has a bundle-like metric $\mathbf{g}_\mathrm{b}$ of bounded geometry (a Riemannian foliation of bounded geometry).

Case of the simple example



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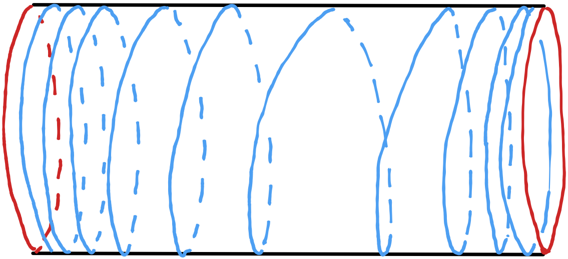

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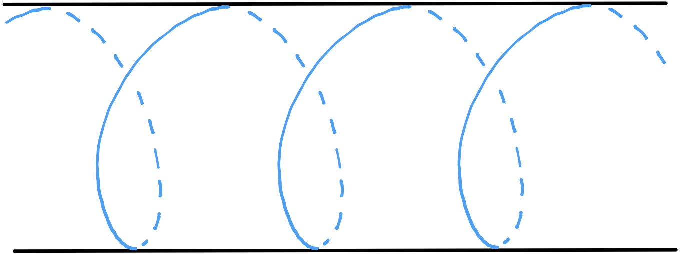


Restriction of the conormal distributions to M^1

- $J(M,L) = \{\, u|_{M^1} \mid u \in I(M,L) \,\} \subset C^\infty(M^1) \equiv C^\infty(\mathring{\mathbf{M}})$.
- $\forall m \in \mathbb{R}$, let

$$J^m(M,L) \equiv \{\, u \in C^\infty(\mathring{\mathbf{M}}) \mid \mathrm{Diff}(M,L) \cdot u \subset \rho^m L^\infty(\mathring{\mathbf{M}}) \,\}\ ,$$

  with the topology defined as before ...
- $J(M,L) = \bigcup_m J^m(M,L)$ (R. Melrose), $\rightsquigarrow$ a LCS.
- The restriction map $R : I(M,L) \to J(M,L)$ is continuous.

Properties of $J(M, L)$

Theorem (Á.-Kordyukov-Leichtnam 2023)

$J(M, L)$ satisfies the same properties as $I(M, L)$.

Theorem (R. Melrose 1996, Á.-Kordyukov-Leichtnam 2023)

The following is a short exact sequence in the category of continuous linear maps between LCSs:

$$0 \to K(M, L) \xrightarrow{\iota} I(M, L) \xrightarrow{R} J(M, L) \to 0 \, .$$

Description of $J(M, L)$

- $\mathbf{g}_{\mathrm{b}}$ of bounded geometry on $\mathring{\mathbf{M}}$ $\rightsquigarrow H^{\pm\infty}(\mathring{\mathbf{M}})$.
- $\boldsymbol{\rho}^{\mu} H^{\pm\infty}(\mathring{\mathbf{M}})$ LCS s.t. $\boldsymbol{\rho}^{\mu} : H^{\pm\infty}(\mathring{\mathbf{M}}) \xrightarrow{\cong} \boldsymbol{\rho}^{\mu} H^{\pm\infty}(\mathring{\mathbf{M}}) \quad (\mu \in \mathbb{R})$.
- $\mu < \mu' \Rightarrow \boldsymbol{\rho}^{\mu} H^{\pm\infty}(\mathring{\mathbf{M}}) \supset \boldsymbol{\rho}^{\mu'} H^{\pm\infty}(\mathring{\mathbf{M}})$.

Theorem (Á.-Kordyukov-Leichtnam 2023)

$$J(M, L) \equiv \bigcup_{\mu} \boldsymbol{\rho}^{\mu} H^{\infty}(\mathring{\mathbf{M}}).$$

Other dual spaces

- $K'(M,L) = K(M,L;\Omega M)'$, $J'(M,L) = J(M,L;\Omega M)'$.
- $K'^{(s)}(M,L)$, $J'^{m}(M,L)$, similar.

Theorem (Á.-Kordyukov-Leichtnam 2023)

1. *$K'(M,L)$ and $J'(M,L)$ satisfy the same properties as $I'(M,L)$.*
2. *The following transposed sequence is exact in the category of continuous linear maps between LCSs:*

$$0 \leftarrow K'(M,L) \xleftarrow{\iota^t} I'(M,L) \xleftarrow{R^t} J'(M,L) \leftarrow 0 .$$

Description of $K'(M, L)$

- Recall:

$$K(M, L; \Omega M) \supset \partial_\rho^m C^\infty(L; \underbrace{\Omega^{-1} NL \otimes \Omega M}_{\Omega L}) \cong C^\infty(L; \Omega L)\ .$$

- $\rightsquigarrow$ a projection (transposing)

$$K'(M, L) \to \partial_\rho^m C^{-\infty}(L) \overset{\partial_\rho^m}{\cong} C^{-\infty}(L)\ .$$

Theorem (Á.-Kordyukov-Leichtnam 2023)

$$K'(M, L) = \prod_{m=0}^{\infty} \partial_\rho^m C^{-\infty}(L; \Omega L) \cong \prod_{m=0}^{\infty} C^{-\infty}(L)\ .$$

Description of $J'(M, L)$

Transposing $J(M, L) \equiv \bigcup_{\mu} \rho^{\mu} H^{\infty}(\mathring{\mathbf{M}})$, we get:

Theorem (Á.-Kordyukov-Leichtnam 2023)

$$J'(M, L) \equiv \bigcap_{\mu} \rho^{\mu} H^{-\infty}(\mathring{\mathbf{M}}) \,.$$

Conormal and dual-conormal leafwise currents

- Now M^0 may have several leaves L.
- $I(\mathcal{F}) = I(M, M^0; \Lambda\mathcal{F})$, $K(\mathcal{F})$, $J(\mathcal{F})$, $I'(\mathcal{F})$, $K'(\mathcal{F})$, $J'(\mathcal{F})$ have the same properties as before.
- All of them are topological complexes with $d_{\mathcal{F}}$.
- $\rightsquigarrow$ leafwise cohomologies $H^\bullet I(\mathcal{F})$, $H^\bullet K(\mathcal{F})$, $H^\bullet J(\mathcal{F})$, $H^\bullet I'(\mathcal{F})$, $H^\bullet K'(\mathcal{F})$, $H^\bullet J'(\mathcal{F})$.
- $\rightsquigarrow$ leafwise reduced cohomologies $\bar{H}^\bullet I(\mathcal{F})$, $\bar{H}^\bullet K(\mathcal{F})$, $\bar{H}^\bullet J(\mathcal{F})$, $\bar{H}^\bullet I'(\mathcal{F})$, $\bar{H}^\bullet K'(\mathcal{F})$, $\bar{H}^\bullet J'(\mathcal{F})$.
- $\rightsquigarrow$ $\phi^* = \{\phi^{t*}\}$ on all of them.
- Invariant by leafwise homotopies of flows.
- $\rightsquigarrow$ we can assume $\phi^t = \mathrm{id}$ on M^0.

Description of $K(\mathcal{F})$

- For preserved leaves $L \subset M^0$,

$$\partial_\rho^m C^\infty(L; \Lambda L \otimes \Omega^{-1} NL) \equiv C^\infty(L; \Lambda L) ,$$
$$d_{\mathcal{F}} \equiv d_L - (m+1)\,\eta\wedge , \quad \phi^{t*} \equiv e^{-(m+1)\varkappa_L t} .$$

- $d_{L,-k} = d_L - k\,\eta\wedge \quad (k = m+1 \geq 1)$ (Witten's type perturbation).
- $\rightsquigarrow H^\bullet_{-k}(L), \quad \beta^r_{-k}(L), \quad \chi_{-k}(L) = \chi(L)$.
- Then

$$K(\mathcal{F}) \equiv \bigoplus_L \bigoplus_{k=1}^\infty C^\infty(L; \Lambda L) ,$$
$$d_{\mathcal{F}} \equiv \bigoplus_L \bigoplus_{k=1}^\infty d_{L,-k} , \quad \phi^{t*} \equiv \bigoplus_L \bigoplus_{k=1}^\infty e^{-k\varkappa_L t} .$$

Description of $H^\bullet K(\mathcal{F})$

Corollary (Á.-Kordyukov-Leichtnam 2023)

$$H^\bullet K(\mathcal{F}) = \bar{H}^\bullet K(\mathcal{F}) \equiv \bigoplus_L \bigoplus_{k=1}^{\infty} H^\bullet_{-k}(L)\,, \quad \phi^{t*} \equiv \bigoplus_L \bigoplus_{k=1}^{\infty} e^{-k\varkappa_L t}\,.$$

- Renormalizing, it makes some sense to define:

$$L_{\mathrm{dis},K}(\phi) = \sum_{\varkappa_L t>0} \chi(L) \sum_{k=1}^{\infty} e^{-k\varkappa_L t} = \sum_{\varkappa_L t>0} \frac{\chi(L)}{e^{\varkappa_L t} - 1} \in C^\infty(\mathbb{R}^\times)\,.$$

Description of $H^\bullet K'(\mathcal{F})$

Similarly, using that $H^r_m(L)' \cong H^{\dim(L)-r}_{-m}(L)$:

Corollary (Á.-Kordyukov-Leichtnam 2023)

$$H^\bullet K'(\mathcal{F}) \equiv \bar{H}^\bullet K'(\mathcal{F}) \equiv \bigoplus_L \prod_{k=0}^{\infty} H^\bullet_k(L)\,, \quad \phi^{t\,*} \equiv \bigoplus_L \prod_{k=0}^{\infty} e^{k\varkappa_L t}\,.$$

- Renormalizing again, it makes some sense to define:

$$L_{\mathrm{dis},K'}(\phi) = \sum_{\varkappa_L t<0} \chi(L) \sum_{k=0}^{\infty} e^{k\varkappa_L t} = \sum_{\varkappa_L t<0} \frac{\chi(L)}{1-e^{\varkappa_L t}} \in C^\infty(\mathbb{R}^\times)\,.$$

Contribution from $H^\bullet K(\mathcal{F})$ and $H^\bullet K'(\mathcal{F})$

- $\rightsquigarrow$ it makes some sense to define:

$$L_{\mathrm{dis},K,K'}(\phi) = L_{\mathrm{dis},K}(\phi) + L_{\mathrm{dis},K'}(\phi) = \sum_L \frac{\chi(L)}{|e^{\varkappa_L t} - 1|} \in C^\infty(\mathbb{R}^\times) .$$

- $\frac{1}{|e^{\varkappa_L t}-1|}$ can be extended to distributions on $\mathbb{R}$, but we continue writing $\frac{1}{|e^{\varkappa_L t}-1|}$ in $C^{-\infty}(\mathbb{R})$ (for clarity and simplicity).

Description of $\bar{H}^\bullet J(\mathcal{F})$

- $d_{\mathcal{F}}\boldsymbol{\rho} = \boldsymbol{\rho}\boldsymbol{\eta} \quad \rightsquigarrow d_{\mathcal{F},\mu} = d_{\mathcal{F}} + \mu\boldsymbol{\eta}\wedge = \boldsymbol{\rho}^{-\mu} d_{\mathcal{F}} \boldsymbol{\rho}^{\mu}$ on $H^{\pm\infty}(\mathring{\mathbf{M}}; \Lambda\mathcal{F})$ (leafwise Witten's perturbation).
- $\phi_\mu^{t\,*} = \boldsymbol{\rho}^{-\mu}\phi^{t\,*}\boldsymbol{\rho}^{\mu}$ on $H^{\pm\infty}(\mathring{\mathbf{M}}; \Lambda\mathcal{F})$.
- $\rightsquigarrow H_\mu^\bullet H^{\pm\infty}(\mathcal{F})$, $\bar{H}_\mu^\bullet H^{\pm\infty}(\mathcal{F})$ with $\phi_\mu^* = \{\phi_\mu^{t\,*}\}$.
- For $\mu < \mu'$, the following diagram is commutative:

$$\begin{array}{ccc}
(\boldsymbol{\rho}^{\mu'} H^{\pm\infty}(\mathring{\mathbf{M}}; \Lambda\mathcal{F}), d_{\mathcal{F}}) & \xrightarrow{\text{inclusion}} & (\boldsymbol{\rho}^{\mu} H^{\pm\infty}(\mathring{\mathbf{M}}; \Lambda\mathcal{F}), d_{\mathcal{F}}) \\
\boldsymbol{\rho}^{\mu'} \uparrow \cong & & \cong \uparrow \boldsymbol{\rho}^{\mu} \\
(H^{\pm\infty}(\mathring{\mathbf{M}}; \Lambda\mathcal{F}), d_{\mathcal{F},\mu'}) & \xrightarrow{\boldsymbol{\rho}^{\mu'-\mu}} & (H^{\pm\infty}(\mathring{\mathbf{M}}; \Lambda\mathcal{F}), d_{\mathcal{F},\mu})
\end{array}$$

Description of $\bar{H}^\bullet J(\mathcal{F})$

$$J(\mathcal{F}) = \bigcup_\mu \rho^\mu H^\infty(\mathring{\mathbf{M}}; \Lambda\boldsymbol{\mathcal{F}}) \equiv \varinjlim H^\infty(\mathring{\mathbf{M}}; \Lambda\boldsymbol{\mathcal{F}}) ,$$

$$d_{\boldsymbol{\mathcal{F}}} \equiv \varinjlim d_{\boldsymbol{\mathcal{F}},\mu} , \quad \phi^{t\,*} \equiv \varinjlim \boldsymbol{\phi}_\mu^{t\,*} \quad (\mu \downarrow -\infty) .$$

Corollary (Á.-Kordyukov-Leichtnam 2023)

$$\bar{H}^\bullet J(\mathcal{F}) \equiv \varinjlim \bar{H}_\mu^\bullet H^\infty(\mathring{\boldsymbol{\mathcal{F}}}) , \quad \phi^{t\,*} \equiv \varinjlim \boldsymbol{\phi}_\mu^{t\,*} .$$

Description of $\bar{H}^\bullet J'(\mathcal{F})$

$$J'(\mathcal{F}) = \bigcap_\mu \rho^\mu H^{-\infty}(\mathring{\mathbf{M}}; \Lambda\boldsymbol{\mathcal{F}}) \equiv \varprojlim H^{-\infty}(\mathring{\mathbf{M}}; \Lambda\boldsymbol{\mathcal{F}}) ,$$

$$d_{\boldsymbol{\mathcal{F}}} \equiv \varprojlim d_{\boldsymbol{\mathcal{F}},\mu} , \quad \phi^{t\,*} \equiv \varprojlim \boldsymbol{\phi}_\mu^{t\,*} \quad (\mu \uparrow \infty) .$$

Corollary (Á.-Kordyukov-Leichtnam 2023)

$$\bar{H}^\bullet J'(\mathcal{F}) \equiv \varprojlim \bar{H}_\mu^\bullet H^{-\infty}(\mathring{\boldsymbol{\mathcal{F}}}) , \quad \phi^{t\,*} \equiv \varprojlim \boldsymbol{\phi}_\mu^{t\,*} .$$

Contributions from $\bar{H}^\bullet J(\mathcal{F})$ and $\bar{H}^\bullet J'(\mathcal{F})$

- We proceed like in the case without preserved leaves, using $d_{\mathring{\mathcal{F}},\mu}$ and ϕ_μ^{t*} on $H^\infty(\mathring{\mathbf{M}};\Lambda\mathring{\mathcal{F}})$, and using the bounded geometry.
- $\rightsquigarrow$ smoothing operators

$$P_{\mu,f} = \int_{\mathbb{R}} \phi_\mu^{t*} \cdot f(t)\,dt \circ \Pi_{\mathring{\mathcal{F}},\mu}\ ,$$

$$P_{\mu,u,f} = \int_{\mathbb{R}} \phi_\mu^{t*} \cdot f(t)\,dt \circ e^{-u\Delta_{\mathring{\mathcal{F}},\mu}}\ .$$

- $\rightsquigarrow$ C^∞ Schwartz kernels $K_{\mu,f}(p,q)$ and $K_{\mu,u,f}(p,q)$.
- But now $P_{\mu,f}$ and $P_{\mu,u,f}$ are not of trace class ($\mathring{\mathbf{M}}$ is not compact).

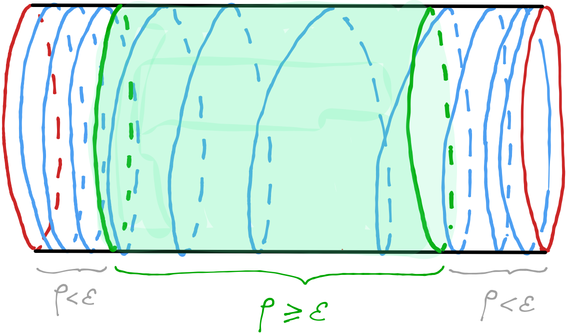
Contributions from $\bar{H}^\bullet J(\mathcal{F})$ and $\bar{H}^\bullet J'(\mathcal{F})$

- However $P_{\mu,u,f}$ has a b-trace:

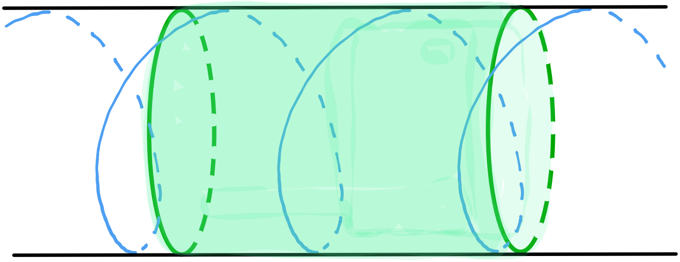
$$
{}^{\mathrm{b}}\mathrm{Tr} P_{\mu,u,f} = \lim_{\epsilon \downarrow 0} \left(\int_{\rho(p)\geq\epsilon} \operatorname{tr} K_{\mu,u,f}(p,p) - \ln\epsilon \cdot C \right) ,
$$

where C is determined by this convergence (Melrose 1993).

Case of the simple example



Case of the simple example



Contributions from $\bar{H}^\bullet J(\mathcal{F})$ and $\bar{H}^\bullet J'(\mathcal{F})$

Theorem (Á.-Kordyukov-Leichtnam 2023)

$$\lim_{u\downarrow 0} {}^{\mathrm{b}}\mathrm{Str}\, P_{\mu,u,f} = {}^{\mathrm{b}}\chi_\omega(\mathring{\mathcal{F}})\, f(0) + \sum_c \ell(c) \sum_{k\neq 0} \epsilon_c(k\ell(c))\, f(k\ell(c))\,.$$

However ${}^{\mathrm{b}}\mathrm{Tr}[A,B] \neq 0$ (the b-trace is not a trace), $\rightsquigarrow \frac{d}{du}{}^{\mathrm{b}}\mathrm{Str}\, P_{\mu,u} \neq 0$.

Theorem (Á.-Kordyukov-Leichtnam 2023)

If $\dim \mathcal{F}$ *is even, we can choose* η *and* $\mathbf{g}_{\mathrm{b}}$ *so that*

$$\langle Z_\mu, f\rangle = \lim_{u\uparrow\infty,\ v\downarrow 0} \left({}^{\mathrm{b}}\mathrm{Str}\, P_{\mu,u,f} - {}^{\mathrm{b}}\mathrm{Str}\, P_{\mu,v,f}\right)$$

defines a tempered distribution on $\mathbb{R}$, *and* $Z_\mu \to 0$ *as* $\mu \to \pm\infty$.

Contributions from $\bar{H}^\bullet J(\mathcal{F})$ and $\bar{H}^\bullet J'(\mathcal{F})$

- $\rightsquigarrow$ it makes sense to define

$$\langle L_{\mathrm{dis},J}(\phi), f\rangle = \lim_{\mu\uparrow\infty}\lim_{u\uparrow\infty} {}^{\mathrm{b}}\mathrm{Str}\, P_{\mu,u,f}\ ,$$
$$\langle L_{\mathrm{dis},J'}(\phi), f\rangle = \lim_{\mu\downarrow-\infty}\lim_{u\uparrow\infty} {}^{\mathrm{b}}\mathrm{Str}\, P_{\mu,u,f}\ .$$

- $\rightsquigarrow$ $L_{\mathrm{dis},J}(\phi) = L_{\mathrm{dis},J'}(\phi) = {}^{\mathrm{b}}\chi_\omega(\mathring{\mathcal{F}})\,\delta_0 + \sum_c \ell(c) \sum_{k\neq 0} \epsilon_c(k\ell(c))\,\delta_{k\ell(c)}$.
- Actually, they can be considered as the same contribution from $I(\mathcal{F}) \cap I'(\mathcal{F}) = C^\infty(M;\Lambda\mathcal{F})$.
- $\rightsquigarrow$ $L_{\mathrm{dis},J,J'}(\phi) = L_{\mathrm{dis},J}(\phi) = L_{\mathrm{dis},J'}(\phi)$.

Trace formula

Theorem (Á.-Kordyukov-Leichtnam 2023)

The above short exact sequences induce short exact sequences

$$0 \to H^\bullet K(\mathcal{F}) \xrightarrow{\iota_*} \bar{H}^\bullet I(\mathcal{F}) \xrightarrow{R_*} \bar{H}^\bullet J(\mathcal{F}) \to 0 \;,$$
$$0 \leftarrow H^\bullet K'(\mathcal{F}) \xleftarrow{\iota_*^t} \bar{H}^\bullet I'(\mathcal{F}) \xleftarrow{R_*^t} \bar{H}^\bullet J'(\mathcal{F}) \leftarrow 0 \;.$$

- $\rightsquigarrow L_{\mathrm{dis}}(\phi) = L_{\mathrm{dis},I,I'}(\phi) = L_{\mathrm{dis},K,K'}(\phi) + L_{\mathrm{dis},J,J'}(\phi)$.

Theorem (Á.-Kordyukov-Leichtnam 2023)

$$L_{\mathrm{dis}}(\phi) = \sum_L \frac{\chi(L)}{|e^{\varkappa_L t} - 1|} + {}^{\mathrm{b}}\chi_\omega(\mathring{\mathcal{F}})\,\delta_0 + \sum_c \ell(c) \sum_{k\in\mathbb{Z}^\times} \epsilon_c(k)\,\delta_{k\ell(c)} \;.$$

Comparison with Weil's explicit formula when $\dim \mathcal{F} = 2$

- The trace formula on $\mathbb{R}^\times$ becomes:

$$1 - (\text{“distributional trace” of } \phi^* \text{ on degree } 1) + 1 = \sum_L \frac{\chi(L)}{|e^{\varkappa_L t} - 1|} + \sum_c \ell(c) \sum_{k \in \mathbb{Z}^\times} \epsilon_c(k)\, \delta_{k\ell(c)} \,.$$

- Weil's explicit formula on $\mathbb{R}^\times$:

$$1 - \sum_{\rho \in \hat{\zeta}^{-1}(0),\ \Re\rho \geq 0} e^{t\rho} + e^t = \frac{1}{1 - e^{-2t}}\, 1_{t>0} + \frac{e^t}{1 - e^{2t}}\, 1_{t<0} + \sum_p \log p \sum_{k \in \mathbb{Z}^+} \left(\delta_{k \log p} + p^{-k} \delta_{-k \log p} \right) \,.$$

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Final comment

- Deninger's programme contains the idea of an interpretation of Weil's explicit formula as a Lefschetz trace formula for a flow.
- It is a rough approximation of the arithmetic reality. But such an interpretation should exist and it would be interesting to reach it.
- We believe that the tools and methods used in our proof might be of interest for such a goal, though it's still far from being achieved.

Thank you very much for your attention!