Geometric formality and foliations

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Overview

Geometrically formal manifolds
Introduction and definition

- Let (M, g) be a Riemannian manifold (assume closed, oriented, connected). The wedge product of differential forms induces the cup product in cohomology:

$$\begin{aligned} H^r(M) \times H^s(M) &\rightarrow H^{r+s}(M) \\ ([\alpha],[\beta]) &\mapsto [\alpha \wedge \beta] \end{aligned}$$

  which works because the wedge product preserves closed forms, and an exact form wedge a closed form is exact.
- The Hodge theorem tells us that each cohomology class $[\alpha]$ contains a unique harmonic form $\widetilde{\alpha}$, where $d\widetilde{\alpha} = 0$, $d * \widetilde{\alpha} = 0$, and this form is the one in the class with the smallest L^2-norm. So in all cases the wedge product of two harmonic forms is a closed form.

Introduction and definition, continued

- It is natural to ask whether the wedge product of two harmonic forms is a in fact also harmonic. The metric g is called **geometrically formal** if the wedge product of any two harmonic forms is harmonic.
- D. Kotschick introduced this notion in 2000/2001 [4], and he and others have shown that the geometrically formal condition places many conditions on the geometry and topology of the manifold.

Some results about formality

- If (M, g) is formal, the pointwise inner product (α, β) of any two harmonic k-forms α and β is constant. In particular, harmonic forms have constant length.

Proof.

$*\beta$ is also harmonic ($*$ commutes with Δ), $\alpha \wedge *\beta = (\alpha, \beta)\, dV$, which must be $(\text{const})\, dV$. □

- Every compact symmetric space is formal.

Proof.

Harmonic forms = invariant forms, and the wedge product of invariant forms is invariant. □

- Remark: locally symmetric spaces might have no formal metrics (e.g. hyperbolic surfaces).

Some results about formality, continued

- Every metric on a rational homology sphere is formal.

Proof.

Only harmonic forms are constants, constant*(volume form). □

- Every manifold that is not a rational homology sphere has metrics that are not formal.

Proof.

Bochner formula gives curvature condition that can always be violated by local change of metric. □

Some results about formality, continued

- Formal metrics on closed surfaces:
  - Any metric on S^2
  - No metric on surfaces of genus ≥ 2 (harmonic one-forms must vanish at points due to Hopf index theorem)
  - Only flat metrics on T^2 are formal (Kotschick proof: Bochner formula gives nonpositivity condition on Gauss curvature; Gauss-Bonnet theorem forces Gauss curvature to be zero)
- Kotschick results: If (M, g) is formal, the (real) Betti numbers satisfy:
  - $b_k(M) \leq b_k(T^n)$
  - if $n = 4m$, $b_{2k}^{\pm}(M) \leq b_{2k}^{\pm}(T^n)$
  - $b_1(M) \neq n - 1$
  - if $b_1(M) = n$, $(M, g) \cong (T^n, \text{flat metric})$.

Some results about formality, continued

- (Kotschick, 2001) There exists a formal metric on M^3 iff M fibers over S^1.
- (Kotschick, 2001) There exists a formal metric on $M^{\leq 4}$ iff M has the real cohomological algebra of a compact symmetric space.
- (C. Bär, 2015 - relationship with Hopf product conjecture that $S^2 \times S^2$ has no metric with sec > 0)
  - If $\left(M^4, g\right)$ is formal and has $\sec\left(g\right) > 0$, then $M \cong S^4$ or $M \cong \mathbb{C}P^2$.
  - If a metric on $S^2 \times S^2$ has the property that a harmonic 2-form α that is not "not that far" from being constant, then Hopf conjecture holds for that metric.

Formality and Foliations
Using the foliation perspective

Lemma

The set of smooth sections of the kernel of a closed p-form ω on a smooth manifold M spans the tangent bundle of a smooth (possibly singular) foliation.

Using the foliation perspective, continued

Proof.

Suppose the vector fields X_0, X_1 are sections of $\ker\omega$. Then for any vector fields $X_2, ..., X_{p+1}$, we have

$$
\begin{aligned}
0 &= d\omega\left(X_0, X_1, ..., X_{p+1}\right) \\
&= \sum_{j=0}^{p+1} (-1)^j X_j \omega\left(X_0, ..., \widehat{X}_j, ..., X_{p+1}\right) \\
&\quad + \sum_{i<j} (-1)^{i+j} \omega\left(\left[X_i, X_j\right], X_0, ..., \widehat{X}_i, ..., \widehat{X}_j, ..., X_{p+1}\right) \\
&= -\omega\left(\left[X_0, X_1\right], X_2, ..., X_{p+1}\right)
\end{aligned}
$$

since all other terms are automatically zero. Thus, $[X_0, X_1] \lrcorner \omega$ is zero also, so that $\ker\omega$ is involutive. □

Using the foliation perspective, continued

Lemma

(Also in [1, Lemma 6.1] and in [2])
Let M be a closed Riemannian manifold. There exists a harmonic one-form α of constant length if and only if there exists a codimension one minimal Riemannian foliation on M.

Using the foliation perspective, continued

Proof.

Suppose that there exists a codimension one minimal Riemannian foliation on M with characteristic form β, which has constant length. Then from Rummler's formula, $d\beta = -\kappa \wedge \beta + \varphi_0$, but κ is zero since the foliation is minimal, and $\varphi_0 = 0$ since the orthogonal foliation has dimension 1 (and is thus integrable). As a bonus, it is geodesic, and we get that $d(*\beta) = 0 \wedge *\beta + 0$ from Rummler's formula. Thus $\alpha = *\beta$ is a harmonic one-form of constant length.

Conversely, Suppose we have a harmonic 1-form on M with constant length. After multiplying by a constant, we get a characteristic form α of the flow by $\alpha^{\#}$. Then, since $d\alpha = 0$, the mean curvature of the flow is zero, and the normal bundle is integrable. Since the flow is geodesic, the normal foliation is Riemannian and has characteristic form $*\alpha$, and it is minimal from Rummler's formula because $d * \alpha = 0$. □

Using the foliation perspective, continued

Lemma

If α, β are nonzero harmonic one-forms of constant length on a closed Riemannian manifold that are linearly independent, then the distribution $\ker(\alpha \wedge \beta)$ *is the tangent bundle of a Riemannian foliation if and only if the pointwise inner product (α, β) is a basic function for that foliation.*

Using the foliation perspective, continued

Here is a simple proof that the only formal metrics on T^2 are the flat ones.

Proof.

Suppose (T^2, g) is formal. Then there exists a one-form α that is harmonic and has constant length. Then $*\alpha$ is also a harmonic one-form and has constant length, and $\alpha \wedge *\alpha = (\alpha, \alpha)\, dV$, and so they are independent, and in fact $(\alpha, *\alpha)\, dV = \alpha \wedge *^2\alpha = \pm\alpha \wedge \alpha = 0$, so they are perpendicular. So $\ker \alpha$ and $\ker(*\alpha)$ are minimal Riemannian flows (i.e. geodesic Riemannian flows, i.e. isometric flows). Thus (T^2, g) is globally symmetric, i.e. constant Gauss curvature 0. □

Transverse formality on Riemannian foliations
Definitions and examples

Definition

Let $(M, \mathcal{F}, g)$ be a Riemannian foliation with bundle-like metric. We say that $(M, \mathcal{F}, g)$ is **transversally formal (or transversally geometrically formal)** if the wedge product of any two basic-harmonic forms is basic-harmonic.

Definition

Let $(M, \mathcal{F}, g)$ be a Riemannian foliation with bundle-like metric. We say that $(M, \mathcal{F}, g)$ is **transversally k-formal (or transversally geometrically k-formal)** if the wedge product of any two basic-harmonic k-forms is basic-harmonic.

Definitions and examples, continued

We note that transverse formality and formality are distinct, in that foliations may satisfy one property but not the other.

Example

In this example, the manifold (M, g) is formal, but the Riemannian foliation $(M, \mathcal{F}, g)$ is not transversally formal. In [5, Theorem 24], Kotschick and Terzic showed that an example introduced by Totaro in [7, Section 1] is a biquotient of $S^3 \times S^3 \times S^3$ (with the standard metric) that is not geometrically formal. This gives a Riemannian foliation (in fact submersion) where the total space is formal, but the foliation is not transversally formal.

Definitions and examples, continued

Example

In this example, the Riemannian foliation $(M, \mathcal{F}, g)$ is transversally formal, but the manifold (M, g) is not formal for any bundle-like metric. Let H be a closed hyperbolic surface, which has first Betti number at least 4. Let $M = H \times S^1$, which has first Betti number at least 5. Then the codimension one foliation of M with leaves of the form $H \times \{\theta\}$ is clearly Riemannian and transversally formal for the product metric. However, by [4, Theorem 6], the first Betti number of a geometrically formal 3-manifold must be 0, 1, or 3; thus the manifold M is not formal for any metric.

What is true: we showed

Lemma

A Riemannian foliation that is 1-formal is also transversally 1-formal.

Transverse twisted formality

- We say that a Riemannian foliation with bundle-like metric is **twisted-formal** if whenever two basic forms α and β are basic twisted-harmonic (i.e. $\widetilde{\Delta}\alpha = \widetilde{\Delta}\beta = 0$), then also $\alpha \wedge \beta$ is basic twisted-harmonic.
- Here the twisted basic Laplacian comes from [3]; it is a modification of the ordinary basic Laplacian that is the square of the corresponding basic Dirac operator, which satisfies Poincaré duality even when the foliation is not taut.
- The twisted basic operators are

$$\begin{aligned} \widetilde{d} &= d - \frac{1}{2}\kappa_b \wedge, \ \widetilde{\delta} = \delta_b - \frac{1}{2}\kappa_b \lrcorner, \\ \widetilde{\Delta} &= \left(\widetilde{d} + \widetilde{\delta}\right)^2. \end{aligned}$$

Transverse twisted formality, continued

- The twisted basic Laplacian and twisted basic cohomology have its advantages over the ordinary basic Laplacian, because $\widetilde{\Delta}$ commutes with the transverse $*$-operator. Thus, Poincaré duality is always satisfied for twisted basic cohomology on Riemannian foliations, and the basic signature of a Riemannian foliation can be defined in general.

But ...

Theorem

If a Riemannian foliation is transversally twisted-formal implies that either the foliation is taut or the twisted cohomology is identically zero.

Proof:

Let α be a basic twisted-harmonic k-form, so that $\widetilde{d}\alpha = 0$ and $\widetilde{\delta}\alpha = 0$. Then $d\alpha = \frac{1}{2}\kappa_b \wedge \alpha$ and $\delta_b\alpha = \frac{1}{2}\kappa_b \lrcorner \alpha$. Since $\overline{*}$ commutes with $\widetilde{\Delta}$, $\overline{*}\alpha$ is also basic twisted-harmonic, and $\alpha \wedge \overline{*}\alpha = |\alpha|^2 \nu$ is twisted-harmonic, where ν is the transverse volume form. Then

$$\delta_b \left(|\alpha|^2 \nu \right) = \frac{1}{2}\kappa_b \lrcorner |\alpha|^2 \nu,$$

or

$$|\alpha|^2 \delta_b(\nu) - d\left(|\alpha|^2\right) \lrcorner \nu = \frac{1}{2}\kappa_b \lrcorner |\alpha|^2 \nu.$$

But ...

Since $\delta_b \nu = \kappa_b \lrcorner \nu$, we have

$$\left(\frac{1}{2} |\alpha|^2 \kappa_b - d\left(|\alpha|^2\right) \right) \lrcorner \nu = 0,$$

iff

$$\widetilde{d}\left(|\alpha|^2\right) = \left(d - \frac{1}{2}\kappa_b \right) \left(|\alpha|^2\right) = 0.$$

If κ_b is not exact, from [3, Theorem 4.2] $\widetilde{H}^0(M, \mathcal{F}) = 0$, so that $|\alpha|^2 = 0$. **QED**.

But ...

The result can actually be improved to:

Theorem

A Riemannian foliation is transversally twisted-formal if and only if the foliation is minimal or the twisted basic cohomology is identically zero.

Proof.

By some additional calculations, if the twisted cohomology is not identically zero, and if the foliation is transversally twisted-formal, then in particular basic twisted-harmonic functions are of the form $Ce^{f/2}$, where C is a constant and $\kappa_b = df$. But then $C^2 e^f$ would also have to be twisted-harmonic, which is not the case unless $f = 0$, where the foliation is minimal. □

Results on transverse formality

- As before, we define a Riemannian foliation to be **transversally formal** if the wedge product of any two basic-harmonic forms is basic-harmonic; here a basic form α is basic-harmonic if $d\alpha = 0$ and $\delta_b \alpha = 0$.

Lemma

If a Riemannian foliation $(M, \mathcal{F})$ is minimal and has involutive normal bundle (e.g. all suspension foliations), then M formal implies transversally formal.

Proof.

This is a consequence of the formula for the basic codifferential and basic Laplacian in terms of the ordinary Laplacian; see [6, Proposition 2.4]. The basic Laplacian is a restriction of the ordinary Laplacian exactly when these two conditions are satisfied. □

Results on transverse formality, continued

Lemma

If M is a 3-manifold, any nontrivial foliation is transversally formal.

Example by Kotschick: Any geometrically formal, closed, oriented 4-manifold admitting a Riemannian metric of nonneg Ricci curvature is diffeomorphic to

1. Mapping torus $M(\varphi)$ where φ is an orientation-preserving isometry of a spherical space form or
2. $\mathbb{RP}^3 \# \mathbb{RP}^3$ with the standard metric.

Results on transverse formality, continued

Lemma

Let $(M, \mathcal{F})$ be a nontaut Riemannian foliation of codimension 2 and $H_b^1 \cong \mathbb{R}$. Then $(M, \mathcal{F})$ is transversally formal for any bundle-like metric.

Proof.

Since the foliation is not taut, $H_b^2 \cong 0$, and any basic-harmonic one-form wedged with itself is zero. □

Results on transverse formality, continued

Lemma

Let $(M, \mathcal{F})$ be a nontaut Riemannian foliation of codimension 2 that has a bundle-like metric that is transversally formal. Then $\dim H_b^1 = 1$.

Proof.

Since the foliation is not taut, $H_b^2 \cong 0$, and any two basic-harmonic one-forms α, β must wedge to zero globally, meaning they are linearly dependent at each point. Then locally we may write $\beta = f\alpha$, and also

$$\begin{aligned} 0 &= d\beta = df \wedge \alpha, \\ 0 &= \delta\beta = df \lrcorner \alpha \end{aligned}$$

Thus, f is constant, and $\dim H_b^1 = 1$. □

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Thank You