A Spectral Sequence for Free Isometric Lie Algebra Actions

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Kraków 28.06.2023

Filtrations of cochain complexes

Let $(C^{\bullet}, d)$ be a cochain complex of vector spaces over $\mathbb{R}$.

Definition

We say that a familly of subspaces $... \subset F^{p+1}C^k \subset F^pC^k \subset ... \subset C^k$ *is a filtration of the complex* $C^{\bullet}$ *if* $d(F^pC^k) \subset F^pC^{k+1}$.

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The spectral sequence associated to a filtration of $C^\bullet$ is given by the folowing set of data:

1. The 0-th page $(E_0^{p,q}, d_0^{p,q})$ given by $E_0^{p,q} = F^pC^{p+q}/F^{p+1}C^{p+q}$ and $d_0^{p,q} : E_0^{p,q} \to E_0^{p,q+1}$ is simply the morphism induced by d.

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2. The r-th page given inductively by:

$$E_r^{p,q} := Ker(d_{r-1}^{p,q})/Im(d_{r-1}^{p,q}) = \frac{\{\alpha \in F^pC^{p+q} \mid d\alpha \in F^{p+r}C^{p+q+1}\}}{F^{p+1}C^{p+q} + d(F^{p-r+1}C^{p+q-1})}$$

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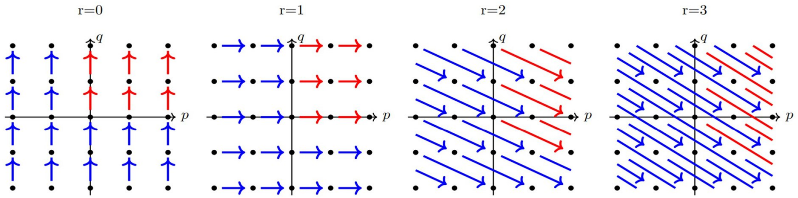
3. The r-th coboundary operator $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ defined again as just the map induced by d (due to the description of the r-th page this has the target specified above and is well defined).

Spectral sequences



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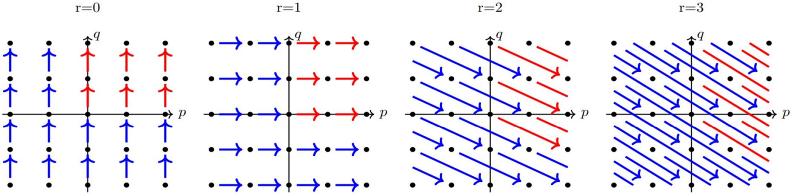

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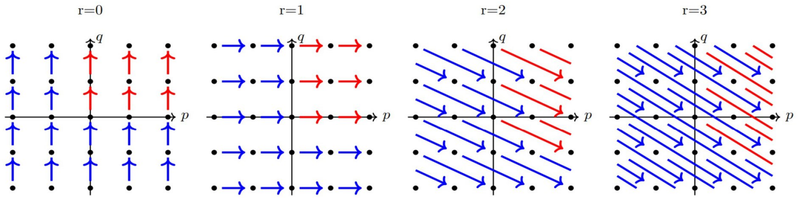

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Proposition

If the only nonzero terms of $E_0^{p,q}$ are in the first quadrant then:

$$H^n(C^\bullet) = \bigoplus E_\infty^{p,q}.$$

Leray-Serre spectral sequence

Let:

$$F \to E \to B,$$

be a smooth locally trivial bundle with fiber F over B. We introduce the following filtration on the de Rham complex $\Omega^\bullet(E)$:

$$F^k\Omega^r(E) := \{\alpha \in \Omega^r(E) \mid \iota_{X_{r-k+1}}...\iota_{X_1}\alpha = 0, \text{ for } X_1, ..., X_{r-k+1} \in \Gamma(TF)\}.$$

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Theorem (Leray-Serre)

The spectral sequence $(E_r^{p,q}, d_r^{p,q})$ associated to the above filtration converges to $H_{dR}^{\bullet}(E)$. Moreover, if $\pi_1(B)$ acts trivially on $H^{\bullet}(F, \mathbb{Z})$, then:

$$E_2^{p,q} \cong H_{dR}^p(B) \otimes H_{dR}^q(F).$$

Case of principal G-bundles

Let G denote a compact connected Lie group with Lie algebra $\mathfrak{g}$. Then we have the following isomorphisms:

$$H^\bullet_{dR}(G) \cong H^\bullet_G(G) \cong H^\bullet(\mathfrak{g}) := H^\bullet(\bigwedge{}^\bullet \mathfrak{g}^*, d_\mathfrak{g}),$$

where,

$$d_\mathfrak{g}\alpha(X_0, ..., X_n) := \sum_{1\leq i<j\leq n} (-1)^{i+j}\alpha([X_i, X_j], X_0, ..., \hat{X}_i, ..., \hat{X}_j, ..., X_n).$$

Corollary

For a principal G-bundle $\pi : E \to B \cong E/G$ the spectral sequence $(E_r^{p,q}, d_r^{p,q})$ satisfies:

$$E_2^{p,q} \cong H^p_{dR}(B) \otimes H^q(\mathfrak{g}).$$

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Let (M^{n+s}, g) be a compact manifold with an isometric locally free action of an s-dimensional connected Lie group G. Then there is a spectral sequence $E_r^{p,q}$ with:

- $E_2^{p,q} = H^p(M/\mathcal{F}, H^q(\mathfrak{g}))$, *where $\mathfrak{g}$ is the Lie algebra of G and $\mathcal{F}$ is the foliation generated by the fundamental vector fields of the action $\xi_1, ..., \xi_s$.*
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Let (M^{n+s}, g) be a compact manifold with a Killing free action of an s-dimensional Lie algebra $\mathfrak{g}$. Then there is a spectral sequence $E_r^{p,q}$ with:

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- $E_r^{p,q}$ *converges to* $H^\bullet_{dR}(M)$.

Lyndon-Hochschield-Serre spectral sequence

Let $\mathfrak{g}$ be an ideal in a Lie algebra $\mathfrak{h}$. Put:

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and denote by $(E_r^{p,q}, d_r^{p,q})$ the spectral sequence associated to this filtration.

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The spectral sequence $(E_r^{p,q}, d_r^{p,q})$ converges to $H^\bullet(\mathfrak{h})$ and satisfies:

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Example

Let $\mathbb{R}$ be a dense subgroup of $\mathbb{T}^2$ then the above spectral sequence converges to $H^\bullet(\mathfrak{g}_{\mathbb{R}^2}) \cong H^\bullet_{dR}(\mathbb{T}^2)$. Despite the orbits not being compact.

Gysin sequence for Sasakian manifolds

Let (M, g, ξ) be a Sasakian manifold with $\eta := g(\xi, \bullet)$. Moreover, let T be the closure of the 1-parameter subgroup of isometries generated by the flow of ξ.

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There is a short exact sequence:

$$0 \to \Omega^{\bullet}(M/\mathcal{F}) \to \Omega^{\bullet}_{T}(M) \xrightarrow{\iota_\xi} \Omega^{\bullet-1}(M/\mathcal{F}) \to 0.$$

which induces a long exact sequence:

$$... \to H^k(M/\mathcal{F}) \to H^k_{dR}(M) \xrightarrow{\iota_\xi} H^{k-1}(M/\mathcal{F}) \xrightarrow{\wedge d\eta} H^{k+1}(M/\mathcal{F}) \to ...$$

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Remark

This is a generalization of the Gysin sequence for circle bundles which itself can be derived from the Leray-Serre spectral sequence.

Commutative case

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Theorem (Raźny)

The spectral sequence $(E^{p,q}_r, d^{p,q}_r)$ converges to $H^\bullet_{dR}(M)$ and satisfies:

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Theorem (Raźny)

The spectral sequence $(E_r^{p,q}, d_r^{p,q})$ *converges to* $H^\bullet_{dR}(M)$ *and satisfies:*

$$E_2^{p,q} \cong H^p(M/\mathcal{F}) \otimes H^q(\mathfrak{g}).$$

Idea of the proof

When the group is abelian we can prove that $\Omega^\bullet_T(M)$ is spanned by $\eta_{i_1} ... \eta_{i_q}\alpha$ for $\alpha \in \Omega^\bullet(M/\mathcal{F})$.

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If in addition the above set of data satisfies the equation:

$$[f, f] + \sum_{k=1}^{s} \xi_k \otimes d\eta_k = 0,$$

where $[f, f]$ *is the Nijenhuis tensor of* f*, then* M *is a* $\mathcal{K}$*-manifold.*

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Definition (Blair)

An (almost) $\mathcal{K}$-structure on a manifold M^{2n+s} is called an (almost) $\mathcal{S}$-structure if $F = d\eta_i$

Application of the spectral sequence

s=0	s=1	general
-	Quasi-K-contact	Almost $\mathcal{K}$
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3. The basic Hodge numbers of $(\mathcal{F}_{Ker(f)}, f)$ are invariant under deformations.

"Right" invariant metric

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Example (Toy)

For a Lie group G consider the left action of G on itself. Then:

1. *The fundamental vector fields* $\{\xi_1, ..., \xi_s\}$ *of the action are right invariant.*
2. *If* $\{X_1, ..., X_s\}$ *are left invariant then* $[X_i, X_j]_e = -[\xi_i, \xi_j]_e$.
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Definition

We say that a metric g on M is virtually right invariant if $g(\xi_i, \xi_j) = const$. *We say that a metric is virtually bi-invariant if it is left invariant and virtually right*

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Proposition

If the action of $\mathfrak{g}$ on (M, g) is free and isometric then M admits a new metric g' which is virtually bi-invariant.

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5. Put $g' = g'_T \oplus g_{\perp}$.

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1. *The cochain complex $(\bigwedge^{\bullet} span(\eta_1, ..., \eta_s), d^{(0,1)})$ is isomorphic to the Lie algebra cochain complex of $\mathfrak{g}$ (with $-d_{\mathfrak{g}}$).*

"Right" invariant forms

The splitting $T\mathcal{F} \oplus (T\mathcal{F})^{\perp}$ induces a splittin on forms with respect to which:

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Corollary

The graded vector subspace $\Omega^{\bullet}_{\mathfrak{g}}(M) \subset \Omega^{\bullet}(M)$ *spanned by differential forms of the form* $\eta_{i_1}...\eta_{i_k}\alpha$ *where* α *is basic, is in fact a subcomplex of* $\Omega^{\bullet}(M)$.

The spectral sequence

Put:

$$F^k\Omega^r_{\mathfrak{g}}(M) := \{\alpha \in \Omega^r_{\mathfrak{g}}(M) \mid \iota_{X_{r-k+1}} ... \iota_{X_1}\alpha = 0, \text{ for } X_1, ..., X_{r-k+1} \in \Gamma(T\mathcal{F})\}.$$

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Theorem (Raźny)

The spectral sequence $(E_r^{p,q}, d_r^{p,q})$ converges to $H^\bullet_{dR}(M)$ and satisfies:

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Theorem

Let $a : \Omega^\bullet(M) \to \Omega^\bullet_{\mathfrak{g}}$ be given by $a(\eta_{i_1,...,\eta_{i_q}}\alpha) = \eta_{i_1,...,\eta_{i_q}}\tilde{a}(\alpha)$ where $\tilde{a}$ is the averaging with respect to $\overline{G}$. Then the splitting $\Omega^\bullet(M) = \Omega^\bullet_{\mathfrak{g}}(M) \oplus Ker(a)$ induces a splitting in cohomology and on the spectral sequence.

Finishing the proof

Proposition

Let (M, g) be a compact Riemannian manifold with a free Killing action of $\mathfrak{g}$. Moreover, let U and V be open saturated subsets. Then the following short sequences of chain complexes is exact:

$$0 \to \Omega_0^\bullet(U \cup V) \to \Omega_0^\bullet(U) \oplus \Omega_0^\bullet(V) \to \Omega_0^\bullet(U \cap V) \to 0.$$

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Proposition

Let (M, g) be a compact Riemannian manifold with a free Killing action of $\mathfrak{g}$. M admits a finite cover such that any intersection of sets of this cover admits a slice.

Low codimension consequences

Theorem

Let (M^{1+s}, g) be a compact oriented Riemannian manifold with a free Killing action of $\mathfrak{g}$. Then:

$$H^{\bullet}(M) \cong H^{\bullet}(\mathfrak{g}) \oplus H^{\bullet-1}(\mathfrak{g})$$

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Example

$\mathbb{R}^s$ *cannot act locally freely on* M^{s+3} *by isometries (under the above conditions).*

Thank you
For your attention.