Open questions concerning foliations in LCK geometry

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Sasakian manifolds, Riemannian foliations, and related topics

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Acknowledgment & Teaser

The talk is based on the recent monograph

L.O., Misha Verbitsky,
Principles of locally conformally Kähler geometry,
arXiv:2208.07188

LCK structures. Definition I

(M, I, g) Hermitian manifold, $\dim_{\mathbb{C}} M = n > 1$, ($I^2 = -1$, integrable), $\omega(x, y) = g(Ix, y)$.

$$d\omega = \theta \wedge \omega, \qquad d\theta = 0$$

(θ is called **Lee form**, after H.-C. Lee, *A kind of even-dimensional differential geometry and its application to exterior calculus*, Amer. J. Math. **65**, (1943), 433–438.)
Usually, we suppose θ non-exact.

Forget the complex structure: (M, ω) is LCS.

Conformal invariance of the notion: if g is LCK, then $e^f g$ is LCK.

Complex submanifolds in LCK are LCK.

LCK structures. Definition II

Let (M, I) be a complex manifold covered by an atlas $\{U_\alpha, \varphi_\alpha\}$ endowed with Kähler forms ω_α, s.t. the transition functions $\varphi_\alpha \varphi_\beta^{-1}$ are homotheties with respect to ω_β.

An LCK form on $(M, \{U_\alpha, \omega_\alpha\})$ is a Hermitian form ω which is conformally equivalent with each ω_α.

LCK structures. Definition III

(M, I) such that its universal cover $\pi : \tilde{M} \to M$ is equipped with a Kähler form $\tilde{\omega}$, and the deck transform group Γ acts on $(\tilde{M}, \tilde{\omega})$ by Kähler homotheties.

Definitions I-III appear in Vaisman's papers, starting with 1976.

Recently extended to complex spaces by Preda-Stanciu.

The **homothety character** is $\chi : \Gamma :\to \mathbb{R}^{>0}, \chi(\gamma) = \frac{\gamma^* \tilde{\omega}}{\tilde{\omega}}$.

Since Γ is a quotient group of $\pi_1(M)$, we can consider χ as a character on $\pi_1(M)$.

The minimal cover of an LCK manifold corresponds to a Γ on which χ is injective (Γ does not contain $\tilde{\omega}$-isometries).

The rank of $\mathrm{Im}(\chi)$ is the **LCK rank of** (M, I, ω).

LCK structures. The weight bundle

Let L be the local system corresponding to the character χ.

Then θ is a flat connection form in L and $\mathrm{Im}(\chi)$ its monodromy.

Call $\alpha \in \Lambda^* \tilde{M}$ **automorphic** if $\gamma^* \alpha = \chi(\gamma)\alpha$.

Automorphic forms on $\tilde{M}$ are identified with L-valued forms on M.

The Morse-Novikov (twisted) cohomology of (M, ω, θ) is the cohomology of the complex $(\Lambda^* M, d_\theta := d - \theta\wedge)$.
It corresponds to the cohomology $H^*(M, L)$ of the local system L and is finite dimensional.

Examples

Almost all (known) non-Kähler compact complex surfaces (Vaisman, Gauduchon-O, Belgun, Brunella).

Hopf manifolds: $(\mathbb{C}^n \setminus 0)/\langle A \rangle$, A being a holomorphic, invertible contraction at 0, linear or non-linear (Kamishima-O, OV).

Some Oeljeklaus-Toma manifolds, generalization in higher dimensions of Inoue surfaces of type S^0 (Oeljeklaus-Toma, Deaconu-Vuletescu).

Kato manifolds, "toric Kato manifolds" – generalizations in higher dimensions of Kato surfaces, i.e. surfaces with global spherical shell (Istrati, Otiman, Pontecorvo, Ruggiero).

Pseudo-effective line bundles. Demailly's result

J.-P. Demailly (*On the Frobenius integrability of certain holomorphic p-forms*, in Complex geometry, 93-98, Springer, 2002):

$L \to M$ holomorphic line bundle

$\Omega \in H^0(\Omega^p(M) \otimes L^{-p})$ holomorphic p-form s.t. the sheaf

$$D := \{v \in TM \mid i_v\Omega = 0\}$$

has rank $(n-p)$ (i.e. Ω is locally identified with a transversal holomorphic volume form of the holomorphic distribution $D \subset TM$.)

Assume that L is **pseudo-effective** (i.e. admits a singular Hermitian metric with curvature equal to a positive, closed current).

If M is compact and Kähler, then the distribution D is integrable.

Pseudo-effective line bundles. Brunella's result

Brunella (Holomorphic dynamical systems, 105-163, L.N.M. **1998**, Springer, 2010): The cotangent sheaf of a rank 1 foliation $F \subset TM$ on a complex manifold is pseudo-effective, unless the closure of any leaf of F is a rational curve.

However, by OV: Strict LCK manifolds are not **uniruled** (cannot be covered by compact families of rational curves).

Hence, by Brunella above: A 1-dimensional holomorphic foliation on a strict LCK manifold has pseudo-effective cotangent sheaf.

Questions on pseudo-effective line bundles

Question 1: Let M be a compact LCK manifold, L a pseudo-effective line bundle, and $\Omega \in H^0(\Omega^p(M) \otimes L^{-p})$ a holomorphic p-form.

Will it follow that the distribution $D := \{v \in TM \mid i_v\Omega = 0\}$ is integrable?

Question 2: Recall:

- Compact LCK manifolds cannot be uniruled.
- Brunella's theorem implies that any rank 1 subsheaf $L \subset TM$ is pseudo-effective.

Will it follow that any compact LCK manifold M does not admit a holomorphic contact structure?

Vaisman manifolds: definition

(M, I, g_M) is LCK

$\nabla^{g_M}\theta = 0$

The condition is not conformally invariant. A Vaisman metric is Gauduchon $(d^*\theta = 0)$.
On compact manifolds, a Vaisman metric, if it exists, is unique up to homothety in its conformal class.

Vaisman manifold: Examples

Diagonal Hopf manifolds $(\mathbb{C}^n \setminus 0)/\langle A \rangle$, $A \in \mathrm{GL}(n, \mathbb{C})$ diagonalizable, with eigenvalues of absolute value > 1;

All compact complex submanifolds of a Vaisman manifold are Vaisman;

Non-Kähler elliptic surfaces;

Some (but not all) small deformations of a compact Vaisman mfd are of Vaisman type.

Non-Vaisman: Non-diagonal Hopf manifolds, Inoue surfaces, Kato manifolds, Oeljeklaus-Toma manifolds, blow-ups of LCK.

Vaisman manifolds: the canonical foliation

$\theta^\sharp$ and $I\theta^\sharp$ are commuting, Killing and real holomorphic vector fields.

Let $\Sigma := \langle \theta^\sharp, I\theta^\sharp \rangle$ be the foliation they generate. It is Riemannian and totally geodesic.

- Regular: the leaf space is a manifold (projective).
- Quasi-regular: compact leaves. The leaf space is an orbifold (projective).

On compact Vaisman, Σ only depends on the complex structure and has at least 1 compact leaf (Tsukada).

Compact complex subvarieties are tangent to Σ.

One has $d^c\theta = \omega - \theta \wedge I\theta$. Moreover, $\Sigma = \mathrm{Ker}(d^c\theta)$ and $d^c\theta$ is positive definite on $\Sigma^\perp$.

A structure theorem for Vaisman manifolds

A compact Vaisman manifold of LCK rank 1 is biholomorphic isometric to a complex manifold obtained by the following receipe:

> Take (S, g_S, η) a compact Sasakian manifold;
>
> Let $(C(S) := S \times \mathbb{R}^{>0}, g := dt \otimes dt + t^2 g_S)$ be its Kähler cone;
>
> Let q be a non-trivial holomorphic homothety of $C(S)$ (along the generators).
>
> Then the compact complex manifold $M = C(S)/\langle q \rangle$ is Vaisman.

Not restrictive since: Let (M, θ, ω) be a compact Vaisman manifold. Then ω can be approximated by a sequence of Hermitian forms which are conformally equivalent to Vaisman metrics of LCK rank 1. (Verbitsky-O)

Topology of compact Vaisman mfds: b_1 is odd, $H^*(M, L) = 0$ (de Leon *et al.* for LCS admitting a metric for which the Lee form is parallel.)

A structure theorem for q-r Vaisman

There exists a negative holomorphic orbifold line bundle L over X, such that M is biholomorphic to a $\mathbb{Z}$-quotient of the space $\mathrm{Tot}^\circ(L)$ of non-zero vectors in L.
The leaves of the canonical foliation are compact, and their preimages in $\mathrm{Tot}^\circ(L)$ coincide with the fibers of L.

Not restrictive since: Any compact Vaisman manifold (M, I) admits a complex deformation (M, I') which is Vaisman and quasi-regular. Moreover, I' can be chosen arbitrarily close to I. (Verbitsky-O)

Vaisman manifolds. First questions

Question 3: Let M be an LCK manifold which has a 1-dimensional transversally Kähler holomorphic foliation. Is it necessarily Vaisman?

Question 4: Let M be an LCK manifold with Lee form θ and Σ the distribution generated by $\theta^\sharp$ and $I\theta^\sharp$. On a Vaisman manifold Σ is integrable, holomorphic and has totally geodesic leaves.

1. Characterize the (compact) LCK manifolds on which Σ is integrable and holomorphic.
2. Characterize the (compact) LCK manifolds on which Σ is integrable and has totally geodesic leaves.

Vaisman manifolds. A splitting problem (I)

$B := TM/\Sigma$ the quotient holomorphic bundle of the canonical foliation on a compact Vaisman manifold.
The exact sequence

$$0 \longrightarrow \Sigma \longrightarrow TM \longrightarrow B \longrightarrow 0 \qquad (1)$$

splits when M is a linear Hopf manifold (because TM is a flat bundle with monodromy $\mathbb{Z}$, acting diagonally, and TM splits as a flat vector bundle.)

Fact: On the Kodaira surface (1) does not split.
(On a Kodaira surface M, B is trivial. Hence, if (1) splits, M is parallelizable and the group of its holomorphic automorphisms is 2-dimensional. But it is known to be 1-dimensional contradiction.)

The same argument implies that (1) does not split when M is a Vaisman manifold of Heisenberg type.

Question 5: Characterize the (compact) Vaisman manifolds for which the exact sequence (1)

$$0 \longrightarrow \Sigma \longrightarrow TM \longrightarrow B \longrightarrow 0$$

splits.

Can (1) split when M is an elliptic fibration over a projective manifold with ample canonical bundle?

Vaisman manifolds. Foliations with compact leaves

Fact: Any smooth rank 1 holomorphic foliation S on a classical Hopf manifold has a compact leaf.
(Follows from a theorem of Baum-Bott: J. Diff. Geometry **7** (1972), 279-342.)

Question 6: Is the same true for all Hopf manifolds?

Question 7: Suppose that S is a rank 1 foliation on a Hopf manifold, not necessarily smooth. Will it follow that S has a leaf with compact closure?

LCK manifolds with potential. Definition

A Kähler cover $\Gamma \longrightarrow (\tilde{M}, \tilde{\omega}) \xrightarrow{\pi} (M, \omega, \theta)$ admits strictly positive and automorphic global potential:

$$\tilde{\omega} = dd^c \varphi, \qquad \gamma^* \varphi = \chi(\gamma) \varphi$$

In this case $\pi^* \theta = d \log \varphi$ and $\pi^* \omega = \frac{dd^c \varphi}{\varphi}$.

There exist LCK manifolds with $\tilde{\omega} = dd^c \varphi$, but φ not automorphic: Oeljeklaus-Toma examples.

There exist LCK manifolds with $\tilde{\omega} = dd^c \varphi$, with φ automorphic, but not positive. In this case (M, I) also admits a positive LCK potential (Verbitsky-O).

Equivalent definitions (on M):

1. $\omega = d_\theta d_\theta^c \varphi_0$, where $\varphi_0 : M \longrightarrow \mathbb{R}^{>0}$.
2. $d^c \theta = \omega - \theta \wedge I\theta$

LCK manifolds with potential. Examples (I)

Vaisman manifolds. Here $\varphi = \|\pi^*\theta\|_{\tilde{\omega}}$.

Non-Vaisman: Non-diagonal Hopf manifolds: $(\mathbb{C}^n \setminus 0)/\langle A \rangle$, $A \in \mathrm{GL}(n, \mathbb{C})$ non-diagonalizable.

If M compact and (M, I, ω) is LCK with potential, then any small deformation (M, I_t) admits LCK metrics with potential. (Verbitsky-O)

Compact LCK not admitting LCK potential: Inoue surfaces (Otiman), Oeljeklaus-Toma manifolds (Kasuya, Otiman).

LCK manifolds with potential. Examples (II)

OV: Essentially, a compact complex manifold of dimension ≥ 3 is LCK with potential if and only if it is a complex submanifold in a Hopf manifold.

"Essentially" refers to:

Up to a deformation to a proper potential (the deck group $\Gamma \cong \mathbb{Z}$, i.e. the LCK rank is 1).

But a compact LCK manifold with potential (M, I, ω, θ) can always be deformed to (M, I, ω', θ') with proper potential.

Sasaki versus LCK with potential

Sasakian manifolds can be defined as level sets of LCK potential on Vaisman manifolds.

Question 8: Is it possible to have an intrinsic definition for the metric contact structure which appears on the level sets of LCK potentials?

The Reeb dynamics on Sasakian manifolds is well understood by now (Rukimbira; using the transversally Kähler structure).

The level sets of the potential on a compact LCK manifold with potential are contact. Hence:

Question 9: Is it possible to prove similar results on the Reeb dynamics for the contact manifolds obtained as level sets of LCK potentials on compact LCK manifolds with potential?

Complex parallelizable manifolds

Recall that a complex manifold is called **complex parallelizable** if its tangent bundle is holomorphically trivial.

Fact: A compact Vaisman manifold cannot be complex parallelizable.

Question 10: Is it possible for a compact LCK manifold with potential to be complex parallelizable?
We expect the answer to be negative.

It is unknown whether a compact complex parallelizable manifold can admit an LCK structure.

Logarithmic foliations (I)

M LCK with potential,

$\tilde{M} \longrightarrow M$ its Kähler $\mathbb{Z}$-covering,

γ generator of the $\mathbb{Z}$-action.

Assume the $\mathbb{Z}$-action on $\tilde{M}$ admits **a logarithm** $\vec{r}$**,** (i.e. a holomorphic, $\mathbb{Z}$-invariant vector field on $\tilde{M}$ s.t. $\gamma = e^{\vec{r}}$)

Then: $\langle \vec{r}, I(\vec{r}) \rangle$ is a holomorphic foliation on M, called **a logarithmic foliation**.

Fact: The logarithm always exists for γ^k, for k sufficiently large; however, the logarithm of γ does not necessarily exist, even for a Vaisman manifold. An example is constructed in the book.

Logarithmic foliations (II)

Recall: An LCK manifold with potential can be obtained as a limit of Vaisman manifolds.

The basic cohomology "seems to be" semicontinuous. Hence:

Question 11: Is the logarithmic foliation on an LCK manifold with potential always taut (a metric exists on M such that all leaves of F are minimal)?

Motivated by: X. Masa (Comment. Math. Helv. **67** (1992), 17-27.): F is taut if and only if the basic cohomology group $H_b^q(M)$ is non-zero, where $q = \operatorname{codim} F$.

In this case, $\dim H_b^q(M) = 1$, and the multiplication in basic cohomology defines a Poincaré-type duality on $H_b^*(M)$.

Question 12: Can a logarithmic foliation on a non-Vaisman LCK manifold with potential admit a transversally Kähler structure? We expect that the answer is negative.

Let M be compact Vaisman, of LCK rank 1, γ the generator of the $\mathbb{Z}$-action. Assume γ has a logarithm, let Ξ be the corresponding logarithmic foliation, let Σ be the canonical foliation, and suppose $\Xi \neq \Sigma$.

Question 13: Will Ξ always admit a transversally Kähler structure? Is Ξ always taut?