Conformal and Minimal Foliations on the Classical Riemannian Symmetric Spaces

- The Method of Eigenfamilies -

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These lecture notes are available at:

www.matematik.lu.se/matematiklu/personal/sigma/slides/2023-06-27-Krakow.pdf

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Co-Workers in Order of Appearance

* 1st wave - Martin Svensson, Anna Sakovich (2006-2014)
* 2nd wave - Stefano Montaldo, Andrea Ratto (2018)
* 3rd wave - Anna Siffert, Marko Sobak, Elsa Ghandour (2020-2023)

Outline

We equip the m-dimensional complex vector space $\mathbb{C}^m$ with the with **complex bilinear form** $\langle\cdot,\cdot\rangle_{\mathbb{C}}$, satisfying

$$\langle z, w\rangle_{\mathbb{C}} = \sum_{k=1}^{m} z_k \cdot w_k,$$

inducing the standard **Euclidean scalar product** $\langle\cdot,\cdot\rangle$ on $\mathbb{R}^m$ as an m-dimension real subspace of $\mathbb{C}^m$ with

$$\langle x, y\rangle = \sum_{k=1}^{m} x_k \cdot y_k.$$

A vector $z \in \mathbb{C}^m$ is said to be **isotropic** if and only if $\langle z, z\rangle_{\mathbb{C}} = 0$.

For two C^1-functions $\phi, \psi : U \subset \mathbb{R}^m \to \mathbb{C}$ we define the symmetric and complex bilinear **conformality operator** κ by $\kappa(\phi,\psi) = \langle\nabla\phi, \nabla\psi\rangle_{\mathbb{C}}$.

If $\phi = u + iv$, then

$$\begin{aligned}\kappa(\phi,\phi) &= \langle\nabla u + i\nabla v, \nabla u + i\nabla v\rangle_{\mathbb{C}}\\ &= (|\nabla u|^2 - |\nabla v|^2) + 2i\cdot\langle\nabla u, \nabla v\rangle.\end{aligned}$$

The complex-valued function $\phi : U \subset \mathbb{R}^m \to \mathbb{C}$ is said to be **horizontally conformal** if and only if $\kappa(\phi,\phi) = 0$ i.e.

$$|\nabla u|^2 = |\nabla v|^2 \quad\text{and}\quad \langle\nabla u, \nabla v\rangle = 0.$$

Definition 1.1 (Harmonic Morphisms in Euclidean $\mathbb{R}^3$)

A complex-valued C^2-function $\phi = u + iv : U \subset \mathbb{R}^3 \to \mathbb{C}$ is said to be a **harmonic morphism** if the composition $f \circ \phi$ with any **holomorphic** function $f : W \subset \mathbb{C} \to \mathbb{C}$ is **harmonic** $(\phi(U) \subset W)$.

Theorem 1.2 (Jacobi (1848))

A complex-valued C^2-function $\phi = u + iv : U \subset \mathbb{R}^3 \to \mathbb{C}$ is a **harmonic morphism** *if and only if it is* **harmonic** *and* **horizontally conformal** *i.e.*

$$\Delta u = 0 = \Delta v, \quad \langle \nabla u, \nabla v \rangle = 0 \quad \text{and} \quad |\nabla u|^2 = |\nabla v|^2.$$

Proof.

$$\Delta(f \circ \phi) = \left[\frac{\partial f}{\partial z}\right] \cdot \Delta \phi + \left[\frac{\partial^2 f}{\partial z^2}\right] \cdot \langle \nabla \phi, \nabla \phi \rangle_{\mathbb{C}} = 0.$$

□

Theorem 1.3 (Jacobi (1848))

Let $f, g : W \subset \mathbb{C} \to \mathbb{C}$ be **holomorphic** *functions, then every local solution $z : U \subset \mathbb{R}^3 \to \mathbb{C}$ to the equation*

$$\langle f(z(x)) \Big[1 + g^2(z(x)), i(1 - g^2(z(x))), 2g(z(x)) \Big], x \rangle_{\mathbb{C}} = 1$$

is a **harmonic morphism**.

Theorem 1.4 (Baird-Wood (1988))

Every harmonic morphism $z : U \to \mathbb{C}$, locally defined in the Euclidean $\mathbb{R}^3$, is obtained this way.

Example 1.5 (The Outer Disc Example)

Let $r \in \mathbb{R}^+$ and choose $g(z) = z$, $f(z) = -1/2irz$ then we yield

$$(x_1 - ix_2)z^2 - 2(x_3 + ir)z - (x_1 + ix_2) = 0$$

with the two solutions

$$z_r^{\pm} : \mathbb{R}^3 \setminus \{(x_1, x_3, x_3) \in \mathbb{R}^3 \,|\, x_1^2 + x_2^2 \leq r^2 \text{ and } x_3 = 0\} \to \mathbb{C},$$

with

$$z_r^{\pm} = \frac{-(x_3 + ir) \pm \sqrt{x_1^2 + x_2^2 + x_3^2 - r^2 + 2irx_3}}{x_1 - ix_2}.$$

Definition 2.1 (Harmonic Morphisms ($m \geq n$))

A C^2-map $\Phi : (M^m, g) \to (N^n, h)$ between Riemannian manifolds is called a **harmonic morphism** if, for any harmonic function $f : U \to \mathbb{R}$ defined on an open subset U of N with $\Phi^{-1}(U)$ non-empty, $f \circ \Phi : \Phi^{-1}(U) \to \mathbb{R}$ is a harmonic function.

Theorem 2.2 (Fuglede (1978), Ishihara (1979))

A C^2-map $\Phi : (M, g) \to (N, h)$ between Riemannian manifolds is a harmonic morphism if and only if it is **harmonic** *and* **horizontally conformal**.

The development of the last 44 years can be traced back at:

∗ **The Bibliography of Harmonic Morphisms** (394 items)

`www.matematik.lu.se/matematiklu/personal/sigma/harmonic/bibliography.html`

Harmonicity (in local coordinates)

For local coordinates x on M and y on N, we have the **non-linear** system

$$\tau(\phi) := \sum_{i,j=1}^{m} g^{ij}\left(\frac{\partial^2\phi^\gamma}{\partial x_i \partial x_j} - \sum_{k=1}^{m}\hat{\Gamma}_{ij}^k \frac{\partial \phi^\gamma}{\partial x_k} + \sum_{\alpha,\beta=1}^{n} \Gamma_{\alpha\beta}^\gamma \circ \phi \cdot \frac{\partial\phi^\alpha}{\partial x_i}\frac{\partial\phi^\beta}{\partial x_j}\right) = 0,$$

where $\phi^\alpha = y_\alpha \circ \phi$ and $\hat{\Gamma}, \Gamma$ are the Christoffel symbols on M, N, resp.

Horizontal Conformality (in local coordinates)

There exists a continuous function $\lambda : M \to \mathbb{R}_0^+$ such that for all $\alpha, \beta = 1, 2, \dots, n$

$$\sum_{i,j=1}^{m} g^{ij}(x)\frac{\partial\phi^\alpha}{\partial x_i}(x)\frac{\partial\phi^\beta}{\partial x_j}(x) = \lambda^2(x) h^{\alpha\beta}(\phi(x)).$$

This is a first order **non-linear** system of $[\binom{n+1}{2} - 1]$ equations.

Our Geometric Motivation:

Theorem 2.3 (Baird, Eells (1981))

Let $\phi : (M^m, g) \to (N^2, h)$ *be a* **horizontally conformal** *map from a Riemannian manifold to a surface. Then* ϕ *is* **harmonic** *if and only if its fibres are* **minimal** *at regular points of* ϕ.

Proof.

$$\tau(\phi) = -\frac{(n-2)}{2} \cdot d\phi(\nabla(\log(\lambda^2))) - (m-n) \cdot d\phi(H)$$

□

The problem is **invariant** under **isometries** on (M, g). If the codomain is a surface $(n = 2)$ then it is also invariant under **conformal changes** $\sigma^2 \cdot h$ of the metric on (N^2, h). This means, at least for local studies, that (N^2, h) can be chosen to be the **standard Euclidean plane** $\mathbb{C}$.

Let (M, g) be a Riemannian manifold, $\mathcal{V} \subset TM$ be an integrable distribution on M and denote by $\mathcal{H} = \mathcal{V}^\perp \subset TM$ its orthogonal complementary distribution. Let $\mathcal{F}$ be the **foliation** tangent to $\mathcal{V}$.

The **second fundamental form** $B^\mathcal{V}$ of $\mathcal{V}$ is given by

$$B^\mathcal{V}(V, W) = \mathcal{H}(\nabla_V W) = \tfrac{1}{2}\mathcal{H}(\nabla_V W + \nabla_W V) \quad (V, W \in \mathcal{V}).$$

The foliation $\mathcal{F}$ is said to be **minimal** if trace $B^\mathcal{V} \equiv 0$ and **totally geodesic** if $B^\mathcal{V} \equiv 0$.

The **second fundamental form** $B^\mathcal{H}$ of $\mathcal{H}$ is defined by

$$B^\mathcal{H}(X, Y) = \tfrac{1}{2}\mathcal{V}(\nabla_X Y + \nabla_Y X) \quad (X, Y \in \mathcal{H}).$$

The foliation $\mathcal{F}$ tangent to $\mathcal{V}$ is said to be **conformal** if there exists a vector field $V \in \mathcal{V}$ such that

$$B^\mathcal{H}(X, Y) = \frac{1}{n} g(X, Y) \otimes V \quad (= g(X, Y) \otimes \operatorname{trace} B^\mathcal{H}) \quad (X, Y \in \mathcal{H}).$$

and $\mathcal{F}$ is said to be **Riemannian** if $V = 0$.

Let (M^m, g) be a Riemannian manifold, $T^{\mathbb{C}}M$ be the complexification of its tangent bundle TM and extend g to a complex-bilinear form on $T^{\mathbb{C}}M$.

Then the **gradient** of a **complex-valued** function $\phi = u + i\,v : (M, g) \to \mathbb{C}$ is the section of $T^{\mathbb{C}}M$ satisfying $\nabla\phi = \nabla u + i\,\nabla v$.

The complex-linear **Laplace-Beltrami operator** τ on (M, g) acts locally on ϕ as

$$\tau(\phi) = \text{div}(\nabla\phi) = \sum_{X \in \text{ONF}} \Big(X^2(\phi) - (\nabla_X X)(\phi)\Big).$$

For two functions $\phi, \psi : (M, g) \to \mathbb{C}$ we have

$$\tau(\phi \cdot \psi) = \tau(\phi) \cdot \psi + 2 \cdot \kappa(\phi, \psi) + \phi \cdot \tau(\psi),$$

where the **conformality operator** κ satisfies $\kappa(\phi, \psi) = g(\nabla\phi, \nabla\psi)$.

The complex-valued function $\Phi : (M, g) \to \mathbb{C}$ is a **harmonic morphism** if and only if

$$\tau(\Phi) = 0 \quad \text{and} \quad \kappa(\Phi, \Phi) = 0.$$

Definition 3.1 (Eigenfunction - Eigenfamily - SG, Sakovich (2008))

Let (M, g) be a Riemennian manifold. Then a complex-valued function $\phi : M \to \mathbb{C}$ is said to be an **eigenfunction** if it is eigen both with respect to the Laplace-Beltrami operator τ and the conformality operator κ i.e. there exist complex numbers $\lambda, \mu \in \mathbb{C}$ such that

$$\tau(\phi) = \lambda \cdot \phi \quad \text{and} \quad \kappa(\phi, \phi) = \mu \cdot \phi^2.$$

A set $\mathcal{E} = \{\phi_i : M \to \mathbb{C} \,|\, i \in I\}$ of complex-valued functions is said to be an **eigenfamily** on M if there exist complex numbers $\lambda, \mu \in \mathbb{C}$ such that for all $\phi, \psi \in \mathcal{E}$ we have

$$\tau(\phi) = \lambda \cdot \phi \quad \text{and} \quad \kappa(\phi, \psi) = \mu \cdot \phi \cdot \psi.$$

Example 3.2 ($\mathbb{R}^{2n}$, $\lambda = 0$, $\mu = 0$)

Let $\mathbb{R}^{2n}$ be the standard **Euclidean space** $\mathbb{R}^{2n} \cong \mathbb{C}^n$ and define $\phi_1, \dots, \phi_n : \mathbb{R}^{2n} \to \mathbb{C}$ by $\phi_j : (x_1, \dots, x_{2n}) \mapsto z_j \cong (x_{2j-1} + i \cdot x_{2j})$. Then the tension field τ and the conformality operator κ on $\mathbb{R}^{2n}$ satisfy

$$\tau(\phi_j) = 0 \quad \text{and} \quad \kappa(\phi_j, \phi_k) = 0.$$

Example 3.3 ($S^{2n-1} = \mathbf{SO}(2n)/\mathbf{SO}(2n-1)$, $\lambda = -(2n-1)$, $\mu = -1$)

Let S^{2n-1} be **the unit sphere** in the standard Euclidean $\mathbb{R}^{2n} \cong \mathbb{C}^n$ and define $\phi_1, \dots, \phi_n : S^{2n-1} \to \mathbb{C}$ by

$$\phi_j : (x_1, \dots, x_{2n}) \mapsto \frac{x_{2j-1} + i \cdot x_{2j}}{\sqrt{x_1^2 + x_2^2 + \cdots x_{2n}^2}}.$$

Then the tension field τ and the conformality operator κ on S^{2n-1} satisfy

$$\tau(\phi_j) = -(2n-1) \cdot \phi_j \quad \text{and} \quad \kappa(\phi_j, \phi_k) = -1 \cdot \phi_j \cdot \phi_k.$$

Example 3.4 ($\mathbb{C}P^n = \mathbf{U}(n+1)/\mathbf{U}(1) \times \mathbf{U}(n)$, $\lambda = -4(n+1)$, $\mu = -4$)

Let $\mathbb{C}P^n$ be the standard n-dimensional **complex projective space**. For a fixed integer $1 \leq \alpha < n+1$ and $1 \leq j \leq \alpha < k \leq n+1$ define the function $\phi_{jk} : \mathbb{C}P^n \to \mathbb{C}$ by

$$\phi_{jk} : [z_1, \dots, z_{n+1}] \mapsto \frac{z_j \cdot \bar{z}_k}{z_1 \cdot \bar{z}_1 + \dots + z_{n+1} \cdot \bar{z}_{n+1}}.$$

Then the tension field τ and the conformality operator κ on $\mathbb{C}P^n$ satisfy

$$\tau(\phi_{jk}) = -\,4(n+1) \cdot \phi_{jk} \quad \text{and} \quad \kappa(\phi_{jk}, \phi_{lm}) = -\,4 \cdot \phi_{jk} \cdot \phi_{lm}.$$

Note that

$$\#(\{\phi_{jk} \mid j \leq \alpha < k\}) = \alpha \cdot (n+1-\alpha).$$

Lemma 3.5 (SG, Sakovich (2008) + SG, Ghandour (2021))

Let $z_{j\alpha} : \mathbf{U}(n) \to \mathbb{C}$ be the complex-valued matrix elements of the standard representation of the unitary group $\mathbf{U}(n)$. Then the tension field τ and the conformality operator κ satisfy the following relations

$$\tau(z_{j\alpha}) = -\, n \cdot z_{j\alpha}, \quad \kappa(z_{j\alpha}, z_{k\beta}) = -\, z_{j\beta}\, z_{k\alpha},$$

$$\tau(\bar{z}_{j\alpha}) = -\, n \cdot \bar{z}_{j\alpha}, \quad \kappa(\bar{z}_{j\alpha}, \bar{z}_{k\beta}) = -\, \bar{z}_{j\beta}\, \bar{z}_{k\alpha},$$

$$\kappa(z_{j\alpha}, \bar{z}_{k\beta}) = \delta_{jk} \cdot \delta_{\alpha\beta}.$$

Theorem 3.6 (SG, Sakovich (2008))

If p is a non-zero element of $\mathbb{C}^n$, then the complex n-dimensional vector space

$$\mathcal{E}_p = \{\phi_a : \mathbf{U}(n) \to \mathbb{C} \,|\, \phi_a(z) = \operatorname{trace}(p^t a z^t),\ a \in \mathbb{C}^n\}$$

is an eigenfamily on $\mathbf{U}(n)$.

Lemma 3.7 (SG, Sakovich (2008))

Let $x_{j\alpha} : \mathbf{SO}(n) \to \mathbb{R}$ be the real-valued matrix elements of the standard representation of $\mathbf{SO}(n)$. Then the following relations hold

$$\tau(x_{j\alpha}) = -\frac{(n-1)}{2} \cdot x_{j\alpha},$$

$$\kappa(x_{j\alpha}, x_{k\beta}) = -\frac{1}{2} \cdot (x_{j\beta}x_{k\alpha} - \delta_{jk}\delta_{\alpha\beta}).$$

Theorem 3.8 (SG, Sakovich (2008))

Let $p \in \mathbb{C}^n$ be a non-zero **isotropic** *element i.e. $p_1^2 + p_2^2 + \cdots + p_n^2 = 0$, then the complex n-dimensional vector space*

$$\mathcal{E}_p = \{\phi_a : \mathbf{SO}(n) \to \mathbb{C} \,|\, \phi_a(x) = \operatorname{trace}(p^t a x^t),\ a \in \mathbb{C}^n\}$$

is an eigenfamily on $\mathbf{SO}(n)$.

Theorem 3.9 (SG - Ghandour (2020))

Let (M, g) be a Riemannian manifold and the set of complex-valued functions

$$\mathcal{E} = \{\phi_i : M \to \mathbb{C} \,|\, i = 1, 2, \dots, n\}$$

be a finite **eigenfamily** *i.e. there exist complex numbers $\lambda, \mu \in \mathbb{C}$ such that for all $\phi, \psi \in \mathcal{E}$*

$$\tau(\phi) = \lambda \cdot \phi \quad \text{and} \quad \kappa(\phi, \psi) = \mu \cdot \phi\, \psi.$$

Then the set of **complex homogeneous polynomials of degree** d

$$\mathcal{P}_d(\mathcal{E}) = \{P : M \to \mathbb{C} \,|\, P \in \mathbb{C}[\phi_1, \phi_2, \dots, \phi_n],\ P(\alpha \cdot \phi) = \alpha^d \cdot P(\phi),\ \alpha \in \mathbb{C}\}$$

is an **eigenfamily** *on M such that for all $P, Q \in \mathcal{P}_d(\mathcal{E})$ we have*

$$\tau(P) = (d\,\lambda + d(d-1)\,\mu) \cdot P \quad \text{and} \quad \kappa(P, Q) = d^2 \mu \cdot P\, Q.$$

$$\dim_{\mathbb{C}} \mathcal{P}_d(\mathcal{E}) = \binom{n+d-1}{n}.$$

Theorem 3.10 (SG, Sakovich (2008))

Let (M, g) be a Riemannian manifold and $\mathcal{E} = \{\phi_1, \dots, \phi_n\}$ be a **finite eigenfamily** *of complex-valued functions on M. If $P, Q : \mathbb{C}^n \to \mathbb{C}$ are linearily independent homogeneous polynomials of the same positive degree then the quotient*

$$\Phi = \frac{P(\phi_1, \dots, \phi_n)}{Q(\phi_1, \dots, \phi_n)}$$

is a non-constant **harmonic morphism***, i.e. $\tau(\Phi) = 0$ and $\kappa(\Phi, \Phi) = 0$, on the open and dense subset*

$$\{p \in M \,|\, Q(\phi_1(p), \dots, \phi_n(p)) \neq 0\}.$$

Definition 4.1 (Riemannian Symmetric Space)

A Riemannian manifold (M, g) is said to be a **symmetric space** if to each point $p \in M$ there exists a global isometry $\sigma_p : (M, g) \to (M, g)$ such that $\sigma_p(p) = p$ and the differential $d\sigma_p : T_pM \to T_pM$ at p satisfies

$$d\sigma_p = -\operatorname{id}_{T_pM}.$$

The irreducible Riemannian symmetric spaces were **classified by Élie Cartan** in 1926. They constitute 20 countably infinite families and 24 exceptional single cases. They are quotents of Lie groups and come in **pairs** $(U/K, G/K)$, where U/K is **compact** and G/K is **non-compact**.

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The irreducible Riemannian symmetric spaces were **classified by Élie Cartan** in 1926. They constitute 20 countably infinite families and 24 exceptional single cases. They are quotents of Lie groups and come in **pairs** $(U/K, G/K)$, where U/K is **compact** and G/K is **non-compact**.

$$S^m = \mathbf{SO}(1+m)/\mathbf{SO}(m) \quad * \quad \mathbb{R}H^m = \mathbf{SO}_o(1,m)/\mathbf{SO}(m)$$
$$\mathbb{C}P^m = \mathbf{U}(1+m)/\mathbf{U}(1)\times\mathbf{U}(m) \quad * \quad \mathbb{C}H^m = \mathbf{U}(1,m)/\mathbf{U}(1)\times\mathbf{U}(m)$$
$$\mathbb{H}P^m = \mathbf{Sp}(1+m)/\mathbf{Sp}(1)\times\mathbf{Sp}(m) \quad * \quad \mathbb{H}H^m = \mathbf{Sp}(1,m)/\mathbf{Sp}(1)\times\mathbf{Sp}(m)$$

Theorem 4.2 (SG, Svensson (2009))

Let (M, g) be an irreducible Riemannian **symmetric space** *other than the* **compact** $G_2/\mathbf{SO}(4)$ *or its* **non-compact** *dual* $G_2^*/\mathbf{SO}(4)$. *Then for each point $p \in M$* **there exists** *a non-constant complex-valued* **harmonic morphism** $\Phi : U \to \mathbb{C}$ *defined on an open neighbourhood U of p. If the space (M, g) is of non-compact type then the domain U can be chosen to be the whole of M.*

Example 4.3 (SG, Sakovich (2008) ** $\lambda \cdot \mu \neq 0$)

Eigenfamilies on the **compact** symmetric spaces

$$\mathbf{SO}(n),\ \mathbf{SU}(n),\ \mathbf{Sp}(n)$$

Example 4.4 (SG, Siffert, Sobak (2020) ** $\lambda \cdot \mu \neq 0$)

Eigenfunctions on the **compact** symmetric spaces

$$\mathbf{SU}(n)/\mathbf{SO}(n),\ \mathbf{Sp}(n)/\mathbf{U}(n),\ \mathbf{SO}(2n)/\mathbf{U}(n),\ \mathbf{SU}(2n)/\mathbf{Sp}(n),$$

Example 4.5 (SG, Ghandour (2022) ** $\lambda = -(m+n)$, $\mu = -2$)

Eigenfamilies on the **compact** real **Grassmannians**

$$\mathbf{SO}(m+n)/\mathbf{SO}(m) \times \mathbf{SO}(n)$$

Example 4.6 (SG, Ghandour (2023) ** $\lambda = -2(m+n)$, $\mu = -2$)

Eigenfamilies on the **compact** complex **Grassmannians**

$$\mathbf{U}(m+n)/\mathbf{U}(m) \times \mathbf{U}(n)$$

Example 4.7 (SG, Ghandour (2023) ** $\lambda = -2(m+n)$, $\mu = -1/2$)

Eigenfamilies on the **compact** quaternionic **Grassmannians**

$$\mathbf{Sp}(m+n)/\mathbf{Sp}(m) \times \mathbf{Sp}(n)$$

Example 4.8 ($\mathbf{U}(m+n)/\mathbf{U}(m)\times\mathbf{U}(n)$ ** $\lambda=-2(m+n),\ \mu=-2$)

For a fixed $1\le\alpha<m+n$ and $1\le j\le\alpha<k\le m+n$ we define the $\mathbf{U}(m)\times\mathbf{U}(n)$-**invariant** functions $\hat\phi_{jk}:\mathbf{U}(m+n)\to\mathbb{C}$ on the unitary group by

$$\hat\phi_{jk}(z)=\sum_{t=1}^{m} z_{jt}\,\bar z_{kt}$$

These induce maps $\phi_{jk}:\mathbf{U}(m+n)/\mathbf{U}(m)\times\mathbf{U}(n)\to\mathbb{C}$ on the quotient space and the tension field τ and the conformality operator κ satisfy

$$\tau(\phi_{jk})=-2(m+n)\cdot\phi_{jk}\quad\text{and}\quad\kappa(\phi_{jk},\phi_{lm})=-2\cdot\phi_{jk}\cdot\phi_{lm}.$$

Note that here we have an **eigenfamily** of complex dimension

$$\#(\{\phi_{jk}\mid j\le\alpha<k\})=\alpha\cdot(m+n-\alpha).$$

Example 4.9 ($\mathbf{SO}(m+n)/\mathbf{SO}(m)\times\mathbf{SO}(n)$ ** $\lambda = -(m+n), \mu = -2$)

Let $p \in \mathbb{C}^{m+n}$ be a non-zero **isotropic** element i.e.

$$p_1^2 + p_2^2 + \cdots + p_{m+n}^2 = 0.$$

Then the $\mathbf{SO}(m)\times\mathbf{SO}(n)$-invariant function $\hat{\phi}_p : \mathbf{SO}(m+n) \to \mathbb{C}$ with

$$\hat{\phi}_p(x) = \sum_{j,\alpha=1}^{m+n} p_j p_k \cdot (\sum_{t=1}^{m} x_{jt}\, x_{kt}),$$

induces an **eigenfunction** $\phi_p : \mathbf{SO}(m+n)/\mathbf{SO}(m)\times\mathbf{SO}(n) \to \mathbb{C}$ on the quotient space with

$$\tau(\phi_p) = -(m+n)\cdot\phi_p \quad \text{and} \quad \kappa(\phi_p,\phi_p) = -2\cdot\phi_p^2.$$

This provides a complex $(m+n-1)$-dimensional family of eigenfunctions on the **real Grassmannian** $\mathbf{SO}(m+n)/\mathbf{SO}(m)\times\mathbf{SO}(n)$.

Definition 5.1 (Proper p-Harmonic Functions)

Let (M,g) be a Riemannian manifold. For a positive integer p, the iterated Laplace-Beltrami operator τ^p is given by

$$\tau^0(\phi) = \phi \quad \text{and} \quad \tau^p(\phi) = \tau(\tau^{(p-1)}(\phi)).$$

We say that a **complex-valued** function $\phi : (M,g) \to \mathbb{C}$ is

(i) **p-harmonic** if $\tau^p(\phi) = 0$, and

(ii) **proper p-harmonic** if $\tau^p(\phi) = 0$ and $\tau^{(p-1)}(\phi)$ does not vanish identically.

∗ **The Bibliography of p-Harmonic Functions** (24 items)

`www.matematik.lu.se/matematiklu/personal/sigma/harmonic/p-bibliography.html`

Theorem 5.2 (SG, Sobak (2020) ** $(\lambda, \mu) \neq 0$)

Let $\phi : (M, g) \to \mathbb{C}$ be a complex-valued function on a Riemannian manifold and $(\lambda, \mu) \in \mathbb{C}^2 \setminus \{0\}$ be such that the tension field τ and the conformality operator κ satisfy

$$\tau(\phi) = \lambda \cdot \phi \quad \textit{and} \quad \kappa(\phi, \phi) = \mu \cdot \phi^2.$$

Then for any positive integer p the non-vanishing function

$$\Phi_p : W = \{x \in M \mid \phi(x) \notin (-\infty, 0]\} \to \mathbb{C}$$

with

$$\Phi_p(x) = \begin{cases} c_1 \cdot \log(\phi(x))^{p-1}, & \textit{if} \;\; \mu = 0,\; \lambda \neq 0 \\ c_1 \cdot \log(\phi(x))^{2p-1} + c_2 \cdot \log(\phi(x))^{2p-2}, & \textit{if} \;\; \mu \neq 0,\; \lambda = \mu \\ c_1 \cdot \phi(x)^{1-\frac{\lambda}{\mu}} \log(\phi(x))^{p-1} + c_2 \cdot \log(\phi(x))^{p-1}, & \textit{if} \;\; \mu \neq 0,\; \lambda \neq \mu \end{cases}$$

is a **proper p-harmonic function***. Here c_1, c_2 are complex coefficients not both zero.*

Outline of the Proof:

Let $\phi : (M, g) \to \mathbb{C}$ be an **eigenfunction** and $f_1 : U \to \mathbb{C}$ be a holomorphic function defined on an open subset U of $\mathbb{C}$ containing the image $\phi(M)$ of ϕ. Then it is a direct consequence of the **chain rule** that the composition $\Phi_1 = f_1 \circ \phi : (M, g) \to \mathbb{C}$ is 1-harmonic if and only if

$$\tau(\Phi_1) = \tau(f_1 \circ \phi) = \kappa(\phi, \phi) \cdot f_1''(\phi) + \tau(\phi) \cdot f_1'(\phi) = 0.$$

$$\tau(\Phi_1) = \tau(f_1 \circ \phi) = \mu\, \phi^2 \cdot f_1''(\phi) + \lambda\, \phi \cdot f_1'(\phi) = 0.$$

Hence Φ_1 is 1-harmonic if and only if f_1 satisfies the complex ODE

$$\mu\, z^2 \cdot f_1''(z) + \lambda\, z \cdot f_1'(z) = 0.$$

Having constructed a 1-harmonic function Φ_1, we can now use this to produce a 2-harmonic function:

Let $\Phi_2 = f_2 \circ \phi$, for some holomorphic function f_2. Then we can make Φ_2 proper 2-harmonic by requiring that it solves the Poisson equation $\tau(\Phi_2) = \Phi_1$. In that case we have $\tau^2(\Phi_2) = \tau(\Phi_1) = 0$.

Applying the **chain rule**, we see that this equation is equivalent to

$$\tau(\Phi_2) = \tau(f_2 \circ \phi) = \kappa(\phi, \phi) \cdot f_2''(\phi) + \tau(\phi) \cdot f_2'(\phi) = f_1(\phi) = \Phi_1,$$

i.e. we want f_2 to satisfy the complex ODE

$$\mu\, z^2 \cdot f_2''(z) + \lambda\, z \cdot f_2'(z) = f_1(z).$$

Having constructed a $(p-1)$-harmonic function Φ_{p-1}, we can now use this to produce a p-harmonic function. Let $\Phi_p = f_p \circ \phi$, for some holomorphic function f_p. Then we can make Φ_p proper p-harmonic by requiring that it solves the Poisson equation $\tau(\Phi_p) = \Phi_{p-1}$. Yet again, employing the **chain rule**, we see that this equation is equivalent to

$$\tau(\Phi_p) = \tau(f_p \circ \phi) = \kappa(\phi, \phi) \cdot f_p''(\phi) + \tau(\phi) \cdot f_p'(\phi) = f_p(\phi) = \Phi_{p-1},$$

i.e. we want f_p to satisfy the complex ordinary differential equation

$$\mu\, z^2 \cdot f_p''(z) + \lambda\, z \cdot f_p'(z) = f_{p-1}(z).$$

□

Let G be a Riemannian Lie group with Lie algebra $\mathfrak{g}$ of left-invariant vector fields on G. If $Z \in \mathfrak{g}$ and $\phi, \psi : U \to \mathbb{C}$ are two complex valued functions defined locally on G then the first and second order derivatives satisfy

$$Z(\phi)(p) = \frac{d}{ds}[\phi(p \cdot \exp(sZ))]\Big|_{s=0},$$

$$Z^2(\phi)(p) = \frac{d^2}{ds^2}[\phi(p \cdot \exp(sZ))]\Big|_{s=0}.$$

The tension field $\tau(\phi)$ and the κ-operator $\kappa(\phi, \psi)$ are given by

$$\tau(\phi) = \sum_{Z \in \text{ONB}} Z^2(\phi) - (\nabla_Z Z)(\phi) \quad \text{and} \quad \kappa(\phi, \psi) = \sum_{Z \in \text{ONB}} Z(\phi) Z(\psi).$$

Let $\mathbf{GL}_n(\mathbb{C})$ be the complex general linear group equipped with its standard Riemannian metric induced by the Euclidean scalar product on the Lie algebra $\mathfrak{gl}_n(\mathbb{C})$ given by

$$g(Z, W) = \mathfrak{Re}\,\mathrm{trace}(Z \cdot W^*).$$

For $1 \leq i, j \leq n$ we shall by E_{ij} denote the element of $\mathfrak{gl}_n(\mathbb{R})$ satisfying

$$(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$$

and by D_t the diagonal matrices

$$D_t = E_{tt}.$$

For $1 \leq r < s \leq n$ let X_{rs} and Y_{rs} be the matrices satisfying

$$X_{rs} = \frac{1}{\sqrt{2}}(E_{rs} + E_{sr}), \quad Y_{rs} = \frac{1}{\sqrt{2}}(E_{rs} - E_{sr}).$$

The unitary group $\mathbf{U}(n)$ is the compact subgroup of $\mathbf{GL}_n(\mathbb{C})$ given by

$$\mathbf{U}(n) = \{z \in \mathbf{GL}_n(\mathbb{C}) |\ z \cdot z^* = I_n\}.$$

The Lie algebra $\mathfrak{u}(n)$ of $\mathbf{U}(n)$ satisfies

$$\mathfrak{u}(n) = \{Z \in \mathbb{C}^{n \times n} |\ Z + Z^* = 0\}$$

and for this we have the canonical orthonormal basis

$$\{Y_{rs}, iX_{rs} |\ 1 \le r < s \le n\} \cup \{iD_t |\ t = 1, \dots, n\}.$$

Lemma 5.3 (SG, Sakovich (2008))

The complex-valued matrix elements $z_{ij} : \mathbf{U}(n) \to \mathbb{C}$ *of* π *satisfy*

$$\tau(z_{j\alpha}) = -n \cdot z_{j\alpha} \quad \text{and} \quad \kappa(z_{j\alpha}, z_{k\beta}) = -z_{j\beta} \cdot z_{k\alpha}.$$

Proof:

It follows directly from the definition of the functions $z_{j\alpha} : \mathbf{U}(n) \to \mathbb{C}$ that if $Z \in \mathfrak{u}(n)$ then the first and second order derivatives satisfy

$$Z(z_{j\alpha}) : z \mapsto e_j \cdot z \cdot Z \cdot e_\alpha^t \ \text{ and } \ Z^2(z_{j\alpha}) : z \mapsto e_j \cdot z \cdot Z^2 \cdot e_\alpha^t.$$

Since $\mathbf{U}(n)$ is compact, we know that if $Z \in \mathfrak{u}(n)$ then

$$\nabla_Z Z = \frac{1}{2} \cdot [Z, Z] = 0.$$

With this at hand, we see that the **Laplace-Beltrami operator** satisfies

$$\begin{aligned} \tau(z_{j\alpha}) &= \sum_{r<s} (Y_{rs}^2 - X_{rs}^2)(z_{j\alpha}) - \sum_t D_t^2(z_{j\alpha}) \\ &= \sum_{r<s} e_j \cdot z \cdot (Y_{rs}^2 - X_{rs}^2) \cdot e_\alpha^t - \sum_t e_j \cdot z \cdot D_t^2 \cdot e_\alpha^t \\ &= -n \cdot z_{j\alpha}, \end{aligned}$$

For the **conformality operator** κ we have

$$\begin{aligned}
\kappa(z_{j\alpha}, z_{k\beta}) &= \sum_{r<s} e_j \cdot z \cdot Y_{rs} \cdot e_\alpha^t \cdot e_\beta \cdot Y_{rs}^t \cdot z^t \cdot e_k^t \\
&\quad - \sum_{r<s} e_j \cdot z \cdot X_{rs} \cdot e_\alpha^t \cdot e_\beta \cdot X_{rs}^t \cdot z^t \cdot e_k^t \\
&\quad - \sum_{t} e_j \cdot z \cdot D_t \cdot e_\alpha^t \cdot e_\beta \cdot D_t^t \cdot z^t \cdot e_k^t \\
&= e_j \cdot z \cdot \Big(\sum_{r<s} Y_{rs} \cdot E_{\alpha\beta} \cdot Y_{rs}^t\Big) \cdot z^t \cdot e_k^t \\
&\quad - e_j \cdot z \cdot \Big(\sum_{r<s} X_{rs} \cdot E_{\alpha\beta} \cdot X_{rs}^t\Big) \cdot z^t \cdot e_k^t \\
&\quad - e_j \cdot z \cdot \Big(\sum_{t} D_t \cdot E_{\alpha\beta} \cdot D_t^t\Big) \cdot z^t \cdot e_k^t \\
&= -e_j \cdot z \cdot E_{\beta\alpha} \cdot z^t \cdot e_k^t \\
&= -\, z_{j\beta} \cdot z_{k\alpha}.
\end{aligned}$$

□

Thank you for your attention !!